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MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 117 No. 5

MEETING OF 1957 MAY 10

Dr W. H. Steavenson, President, in the Chair

The election by the Council of the following Fellows was duly confirmed :—

Bibhu Prasad Dash, Department of Applied Geophysics, Imperial College, London, S.W.7 (proposed by J. M. Bruckshaw);

William Russell Hindmarsh, University Observatory, Oxford (proposed by H. H. Plaskett);

Sudhirendra Nath Saha, Geophysics Department, Imperial College, London, S.W.7 (proposed by J. M. Bruckshaw);

David Vyvyan Thomas, Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex (proposed by P. A. Wayman); and

Neville John Woolf, Department of Astronomy, University of Manchester (proposed by Z. Kopal).

One hundred and fourteen presents were announced as having been received since the last meeting, including :—

G. de Vaucouleurs, *Discovery of the Universe* (presented by the author);

Royal Observatory, Cape of Good Hope, *Cape Photographic Atlas of Southern Galaxies* (presented by the Royal Observatory);

R. A. Buckingham, *Numerical Methods* (presented by Sir Isaac Pitman & Co., Ltd.); and

B. J. Bok and P. F. Bok, *The Milky Way*, 3rd edition (presented by the Harvard and Oxford University Press).

MEETING OF 1957 OCTOBER 11

Dr W. H. Steavenson, President, in the Chair

The election by the Council of the following Fellows was duly confirmed :—

Douglas West Allan, Department of Geodesy and Geophysics, Madingley Road, Cambridge (proposed by E. C. Bullard);

Bart Jan Bok, Mount Stromlo Observatory, Canberra, A.C.T., Australia (proposed by H. H. Plaskett);

Sidney George Coomber, ' Belmont ', Hen Parc Lane, Upper Killay, Swansea, Glam. (proposed by B. Featherstone);

Digby Henry Christ, 744 Warwick Road, Solihull, Warwickshire (proposed by M. C. Johnson);

Vincent Deasy, Ard Faill, Raheny, Dublin, Eire (proposed by H. A. Brück);

George Alfred Harding, Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex (proposed by A. Hunter);

Edgar Alpin Hill, 1 Bayswater Road, Sketty, Swansea, Glam. (proposed by B. Featherstone);

William Aubrey Hill, Highways, Hill Grove, Caswell, Swansea, Glam. (proposed by B. Featherstone);

David Reginald Lynn Jones, Bristol Aeroplane Company, 55 Salisbury Road, Maesteg, Glam. (proposed by G. M. Brown);

Harold Edward Herbert Kirkby-Johnson, 1 Eastfield Road, Royston, Herts. (proposed by H. Thomson);

Muiris MacIonnraic, 53 Mount Prospect Avenue, Clontarf, Dublin, Eire (proposed by H. A. Brück);

John Dennis Pope, Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex (proposed by R. v. d. R. Woolley);

William Henry Thomas, 1 Trinity Place, Swansea, Glam. (proposed by B. Featherstone); and

Clarence Benton Warrenburg, 5835 N 2nd Avenue, Phoenix, Arizona, U.S.A. (proposed by A. Knight).

Two hundred and eighty-four presents were announced as having been received since the last Meeting, including :—

S. K. Banerji, *Earthquakes in the Himalayan Region* (presented by the Indian Association for the Cultivation of Science);

E. M. Burbidge and others, *Synthesis of the Elements in Stars* (presented by the authors);

A. C. Clarke, *The Making of a Moon* (presented by Fredrich Muller, Ltd.);

T. G. Cowling, *Magneto Hydrodynamics* (presented by the author);

Y. Hagihara, *Stability in Celestial Mechanics* (presented by the author);

H. M. Nautical Almanac Office, *Sight Reduction in Tables for Air Navigation*, Vol. 1 (presented by H.M. Nautical Almanac Office);

International Astronomical Union, *Symposium No. IV* (presented by the International Astronomical Union);

W. J. Luyten, *A Catalogue of* 9867 *Stars in the Southern Hemisphere* (presented by the author);

M. K. Munitz, *Space, Time and Creation* (presented by the Free Press);

M. K. Munitz, *Theories of the Universe* (presented by the Free Press);

N. B. Slater, *The Development and Meaning of Eddington's Theory* (presented by the Cambridge University Press); and

H. C. van de Hulst, *Light Scattering by Small Particles* (presented by Chapman and Hall Ltd.).

THE MOON'S LIBRATION IN LONGITUDE

Harold Jeffreys

(Received 1957 June 5)

Summary

Analysis of Yakovkin's results on a possible free libration of the Moon in longitude shows that the motion found agrees in speed and phase with the forced term whose argument is twice the difference of longitude of the perigee and node. If it is interpreted as due to this term

$$\gamma = 0{\cdot}0002049 \pm 0{\cdot}0000009.$$

If the motion is a free one

$$\gamma = 0{\cdot}0002098 \pm 0{\cdot}0000022$$

The results imply values of f close to 0·67 and therefore intermediate between the two groups of determinations made from the annual libration.

Search for the Moon's free libration in longitude had been unsuccessful until recently, and γ, the dynamical ellipticity of the Moon's equator, has been found from the annual libration. The results, when expressed in terms of the ratio f, cluster about 0·5 and 0·8, and no satisfactory explanation has been available. The situation has been completely changed by two important papers of A. A. Yakovkin (1952) and K. Koziel (1949). Yakovkin appears to have detected the free libration, with an amplitude of about 50″. Koziel points out the possible importance of a term whose argument is twice the difference of the longitudes of the node and perigee ; for $f = 0{\cdot}662$ this would have the same period as the free libration, and accordingly it might be large enough to affect the solution. The constants α, β, γ are defined in terms of the moments of inertia by

$$\frac{C-B}{A} = \alpha, \quad \frac{C-A}{B} = \beta, \quad \frac{A-B}{C} = \gamma$$

and $f = \alpha/\beta$. To a very close approximation

$$\alpha + \gamma = \beta.$$

Yakovkin assumes a trial speed of $0^\circ{\cdot}3318$/day and determines a harmonic term from the Kazan observations, dividing them into four intervals according to observers. The results are expressed in the form

$$\text{Libration} = P'' \sin\,(0^\circ{\cdot}3318\,(t - t_0) + p^\circ) \quad \ldots\ldots\ldots \quad (1)$$

where $t_0 =$ J.D.2412000. Uncertainties are standard errors.

	P	p
1910–15	51 ± 11	248 ± 15
1916–31	46 ± 10	206 ± 12
1932–42	60 ± 13	187 ± 13
1938–45	78 ± 16	184 ± 14

A least squares solution is

$$\left.\begin{array}{l} p = 200{\cdot}8 \pm 6{\cdot}7 - (2{\cdot}10 \pm 0{\cdot}62)(t - 1930{\cdot}5)/1 \text{ year} \\ P = 55 \pm 6{\cdot}0 \end{array}\right\} \quad \ldots \quad (2)$$

$\chi^2 = 4{\cdot}4$ on 5 degrees of freedom.

The first question is whether the libration found is actually the free libration or a forced one due to the term mentioned by Koziel. We note that its speed is, in all, $0^\circ{\cdot}3261 \pm 0^\circ{\cdot}0017$/day; and the speed of the forced term would be $0^\circ{\cdot}3287$/day. The phase of the observed libration at 1930·5 would be

$$200^\circ{\cdot}8 \pm 6{\cdot}7 + 14160(0^\circ{\cdot}3318) = 13 \times 360^\circ + 219^\circ{\cdot}1 \pm 6^\circ{\cdot}7 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

That of the forced term would be $212^\circ{\cdot}0$. If the identification with the forced term is correct the comparison contributes 3·4 to χ^2 on 2 degrees of freedom. Hence this identification is perfectly possible. Since a free libration might have had any phase whatever there is strong reason to suppose that it is correct.

We make two solutions, according as the motion is interpreted as free or forced. If it is free it leads to an estimate of γ in the usual way. If it is forced the amplitude will give an equation for γ. Koziel's form for a forced motion can be written

$$\chi = -\frac{2{\cdot}963\,\gamma H \sin(ht+\alpha)}{(h/n)^2 - 2{\cdot}9198\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$n = 13^\circ{\cdot}176$/day. If the motion is free, the denominator vanishes for $h = (0^\circ{\cdot}3261 \pm 0^\circ{\cdot}0017)$/day ; hence

$$\gamma = 0{\cdot}0002098 \pm 0{\cdot}0000022 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For the forced motion Koziel gives (p. 78)

$$H = -0{\cdot}00001165 \text{ radians} = -2{\cdot}403'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in (4), with $h = 0^\circ{\cdot}3287$/day, gives

$$\gamma = 0{\cdot}0002049 \pm 0{\cdot}0000009 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Exact coincidence of speeds would give $\gamma = 0{\cdot}0002131$.

The inclination of the Moon's axis to the ecliptic gives, according to de Sitter,

$$\beta(1 + 0{\cdot}0047f) = 0{\cdot}0006286 \pm 0{\cdot}0000015 (\text{p.e.})$$

whence, with the close approximation $\beta f = \beta - \gamma$ and either above value of γ,

$$\beta = 0{\cdot}0006266 \pm 0{\cdot}0000022 (\text{s.e.}).$$

and

$$f = 1 - \gamma/\beta = 0{\cdot}665 \pm 0{\cdot}0035 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{or } f = 0{\cdot}673 \pm 0{\cdot}0018 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The uncertainty of f still comes mainly from that of γ ; but as there still seems to be some conflict between results about the value of β its uncertainty may be somewhat underestimated. Since the uncertainties of β and γ are nearly independent it seems desirable that they should be given separately and that estimates of γ should cease to be expressed by way of f.

The apparent accuracy of the two solutions for γ far exceeds that of any based on the annual libration, but they seem to be the only possible interpretations of Yakovkin's data. I think that the interpretation as a forced motion is the more likely. Both require a close agreement between the free and forced periods in question, so that this coincidence is equally applicable to both ; but the agreement in phase requires a second coincidence if the motions is free but not if it is forced.

On this interpretation the data still do not separate a free libration ; its amount would be unlikely to exceed 10″.

A possible objection may be that for a close coincidence of periods the first-order theory may not be valid and that higher terms should be taken into account

by something like E. W. Brown's theory of resonance. However an amplitude of 1′ or so in a pendulum motion would produce a change of free period by about 1 part in 10^7, and it seems that any departure from non-linearity must be negligible in comparison with the observational uncertainty.

Koziel, noting that the amplitude of the forced term would theoretically be infinite on a linear theory if $f = 0{\cdot}662$*, maintains that if a trial value of f exceeds this value, any solution found by the method of least squares would still exceed it; and similarly for a trial value less than 0·662. I do not follow this part of his argument, since with a trial value of either 0·5 or 0·8 this term would be negligible in all approximations by corrections linear in the changes of f. But if any use is to be made of the annual term in future it would be desirable to analyse the data in such a way as to give separate determinations of the annual term and one with a speed near $0^{\circ}{\cdot}33$/day. Some of the series of observations that have been used are short enough for a wrong estimate of the amplitude of the latter term to produce an important bias in that of the annual term.

* I get 0·660, presumably owing to a slightly different value of β.

160 *Huntington Road,*
Cambridge:
1957 *June* 4

References

A. A. Yakovkin 1952, *Trans. I.A.U.*, **8**, 231.
K. Koziel, 1949, *Acta Astronomica*, a, **4**, 61–193.

Note added 1957 *August* 1.—Professor Yakovkin informs me that there is a numerical error in the calculation of P, which he has not yet succeeded in correcting. It should be about 100. This would change the estimate of γ, on the assumption that the motion is forced, to 0·0002081, and f would be 0·667. A fresh analysis for the annual term gives $f = 0{\cdot}69 \pm 0{\cdot}02$. A summary has appeared in the Astronomical Circular of the U.S.S.R. Academy, No. 178, 1957 March 12.

THE SECULAR ACCELERATION OF THE MOON, AND THE LUNAR TIDAL COUPLE

C. A. Murray

(Communicated by the Astronomer Royal)

(Received 1957 July 24)

Summary

The discussion of the modern observations of the Sun, Mercury and Venus by Spencer Jones, when taken in conjunction with Jeffreys's theory of tidal friction, is shown to lead to a total dissipation of energy in the oceanic tides which is three times that calculated by Jeffreys in his discussion of tidal data for shallow seas.

The true (negative) acceleration of the Moon corresponding to this value of the dissipation has been included in the expression for the Moon's tabular mean longitude in the *Improved Lunar Ephemeris*. An investigation of the value of this acceleration in ancient times using Hipparchus' eclipse and equinox observations leads to a value twice as large as that given by the modern observations. This result is very satisfactorily confirmed by the observed magnitudes of partial lunar eclipses recorded in the *Almagest*.

It is pointed out that if the acceleration does change, then ephemeris time derived from lunar observations will not be uniform in the Newtonian sense.

---

1. Let the observed correction to the gravitational mean longitude of the Moon be expressed in the form

$$\delta L = a + bT + (q+s)T^2 + B(T) \tag{1}$$

where q is the real (negative) acceleration due to the reaction of the lunar oceanic tidal couple, s is the apparent acceleration due to any secular retardation of the Earth's rotation and $B(T)$ is the fluctuation due to irregularities in the Earth's rotation; a and b are corrections to observationally determined constants of integration.

If N is the lunar oceanic retarding couple, M, m the masses of the Earth and Moon, and c the mean distance between them (all in c.g.s. units), and if we assume for simplicity that the Moon moves in a circular orbit in the plane of the Earth's equator, then Kepler's third law and the principle of conservation of angular momentum give the rate of change of the Moon's orbital angular velocity as

$$-3N\frac{M+m}{Mmc^2}\text{ radian/sec}^2.$$

Thus, if q is expressed in seconds of arc and T in centuries,

$$q = -\frac{3}{2}N\frac{M+m}{Mmc^2}\times 2{\cdot}05\times 10^{24}.$$

Putting $M = 5{\cdot}98\times 10^{27}$, $M/m = 81{\cdot}5$, $c = 3{\cdot}84\times 10^{10}$, we have

$$q = -2{\cdot}88\times 10^{-23}N. \tag{2}$$

2. Jeffreys (1) has pointed out that, in the case of the solar tidal couple N', the corresponding real acceleration of the Sun is observationally quite negligible.

Accordingly the observed correction to the gravitational mean longitude of the Sun may be expressed in the form

$$\delta L' = a' + b'T + (n'/n)\{sT^2 + B(T)\} \tag{3}$$

where n, n' are the mean motions of the Moon and Sun. In this case the apparent departure from a purely gravitational orbit arises solely from non-uniformity in the Earth's rotation.

3. In his paper on the rotation of the Earth, Spencer Jones (2) adopted definite numerical values for a, b and $(q+s)$ in (1) and thus, from observations of the Moon's longitude δL, defined $B(T)$ as an empirical function of the time. By eliminating $B(T)$ from (3), and from similar equations for Mercury and Venus, he determined $(n'/n)s$ from a discussion of observations extending from, roughly, 1680 to 1930.

He actually adopted $q+s = +5''\cdot 22$ and derived $(n'/n)s = +1''\cdot 23 \pm 0''\cdot 04$. Since $n/n' = 13\cdot 37$, his discussion gives

$$q = -11''\cdot 22 \pm 0''\cdot 53. \tag{4}$$

Thus, from (2),

$$N = (3\cdot 9 \pm 0\cdot 2) \times 10^{23} \text{ dyn cm.}$$

Inserting this in Jeffreys's equation for the dissipation (*loc. cit.* p.227), and taking $N/N' = 3\cdot 4$ we have

$$dE/dt = (3\cdot 5 \pm 0\cdot 2) \times 10^{19} \text{ erg/sec.} \tag{5}$$

This is nearly three times the rate of working calculated by Jeffreys from his discussion of tidal data; he remarks, however, that there is probably some dissipation along open shores which he did not include in his calculations.

4. The value of q derived in (4) is of course the correction which has been applied to the theoretical secular acceleration in constructing the *Improved Lunar Ephemeris* (3), as originally proposed by Clemence (4). It is an empirical correction derived from observations extending over the last three centuries. There is no *a priori* reason why it should remain constant over long periods of time, and it is desirable to derive the value which best represents the available ancient observations.

In (1), $sT^2 + B(T)$ is the apparent displacement of the Moon in its orbit due to non-uniformity in the Earth's rotation. It can only be determined in practice from observations of the Sun or planets. (It clearly cannot be determined from observations of the Moon unless q is known *a priori*.) In order to derive a value of q it is thus necessary to have observations of the Moon and Sun (or planets) at as nearly the same epoch as possible.

If δL, $\delta L'$ are corrections to the tabular longitudes of the Moon and Sun at distant epochs T_1, T_2 respectively, the average sidereal secular accelerations ν, ν' over the whole periods T_1, T_2 are defined by

$$\nu = T_1^{-2}\delta L, \quad \nu' = T_2^{-2}\delta L'$$

Thus from (1) and (3), neglecting a, b, a', b'

$$q = \nu - (n/n')\nu' \tag{6}$$

provided that $T_1^{-2}B(T_1) - T_2^{-2}B(T_2)$ is negligible. We assume that this will be the case if T_1 and T_2 do not differ by more than a few years.

5. The observations of Hipparchus in the second century B.C. afford perhaps the most reliable determination of q in ancient times. His series of equinox

observations has been discussed by Fotheringham (5) who derived

$$\nu' = +1''\cdot95 \pm 0''\cdot27 \tag{7}$$

for the correction to Newcomb's sidereal acceleration of the Sun for mean epoch -137. His observation of the solar eclipse of -128 Nov. 20 has also been thoroughly discussed by Fotheringham (6) who derived the following equation of condition for corrections to the sidereal accelerations of Brown's and Newcomb's tables:

$$\nu - \frac{7}{4}\nu' = +2''\cdot10 \pm 0''\cdot55. \tag{8}$$

Combining (7) and (8) we find from (6)

$$q = -20''\cdot6 \pm 3''\cdot2. \tag{9}$$

6. As has been remarked by Jeffreys (*loc. cit.* p. 224), it is also possible to obtain q directly from the magnitudes of lunar eclipses. Apart from the small effects due to differences of topocentric libration, the magnitude as seen from anywhere on the Earth depends only on the geocentric co-ordinates of the Moon and Sun. It is thus independent of U.T.

If λ_0, λ'_0 are the ecliptic longitudes of the Moon and Sun, β_0 the latitude of the Moon, and $\dot\lambda$, $\dot\lambda'$, $\dot\beta$ their respective rates of change at an E.T. t_0, say, sufficiently close to opposition, the relative co-ordinates of the Moon and shadow at time t may be written

$$\lambda - \lambda' + 180^\circ = \lambda_0 - \lambda'_0 + 180^\circ + (\dot\lambda - \dot\lambda')(t - t_0),$$
$$\beta = \beta_0 + \dot\beta(t - t_0).$$

The angular distance between the centres of the Moon and shadow is $\{(\lambda - \lambda' + 180^\circ)^2 + \beta^2\}^{1/2}$ which has the minimum value

$$\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \left|\beta_0 - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}(\lambda_0 - \lambda'_0 + 180^\circ)\right|. \tag{10}$$

Consider the change in this minimum value corresponding to a change qT^2 in the Moon's mean longitude. We have

$$\Delta + \delta\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \left|\beta_0 - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}(\lambda_0 - \lambda'_0 + 180^\circ) + qT^2\left(\frac{\partial\beta}{\partial L} - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}\frac{\partial\lambda}{\partial L}\right)\right|. \tag{11}$$

Now let t_0 be the actual E.T. of opposition; then (11) and (10) become

$$\Delta + \delta\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} |\beta_0 + \delta\beta_0| \tag{12}$$

$$\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} |\beta_0| \tag{13}$$

where β_0 is now the latitude of the Moon at opposition and

$$\delta\beta_0 = qT^2\left(\frac{\partial\beta}{\partial L} - \frac{\dot\beta}{\dot\lambda - \dot\lambda'}\frac{\partial\lambda}{\partial L}\right). \tag{14}$$

For an eclipse to be partial, $|\beta_0|$ must be at least of the order $26'$ whereas, in the eclipses discussed below, the maximum value of $|\delta\beta_0|$ is $6'\cdot2$. Accordingly $|\beta_0| > |\delta\beta_0|$ and we can write

$$\delta\Delta = \pm\left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \delta\beta_0 \tag{15}$$

according as β_0 is positive or negative.

It can readily be verified from numerical formulae given by Cowell (**7**) that

$$\left\{1+\frac{\dot{\beta}^2}{(\dot{\lambda}-\dot{\lambda}')^2}\right\}^{-1/2}\left(\frac{\partial\beta}{\partial L}-\frac{\dot{\beta}}{(\dot{\lambda}-\dot{\lambda}')}\frac{\partial\lambda}{\partial L}\right)\approx\mp 0\cdot 0076(1-0\cdot 075\cos g) \qquad (16)$$

where g is the Moon's mean anomaly and the upper or lower sign is to be taken according as the eclipse is at the ascending or descending node. Also, if σ is the Moon's semi-diameter, then

$$\sigma \approx 942''(1+0\cdot 064\cos g). \qquad (17)$$

If δG is a small change in the magnitude of an eclipse, we have

$$\delta G = -\frac{1}{2\sigma}\delta\Delta \qquad (18)$$

or, combining (14), (15), (16) and (17),

$$\delta G = \pm 4\cdot 0\times 10^{-6}(1-0\cdot 14\cos g)qT^2. \qquad (19)$$

The upper or lower sign in (19) is to be taken according as the Moon is eclipsed after or before passing the node.

7. The observed magnitudes of eleven partial lunar eclipses recorded by Ptolemy in the *Almagest* have been compared with tabular magnitudes by Cowell (**8**) who took as first approximations to the secular accelerations $\nu_c = +4''\cdot 88$, $\nu'_c = +4''\cdot 11$, or from (6) $q_c = -50''$. The observed *minus* tabular magnitudes are given in Table I, together with the values of g and the epochs T measured in centuries from 1800. These have been taken from the tables on pp. 524-7 of Cowell's paper with the exception of the magnitude of eclipse No. 18, for which Fotheringham's corrected value has been taken (**9**).

Table I

Ref. no.	T	g	$\pm 4\times 10^{-6}\, T^2\,(1-0\cdot 14\cos g)$	δG	R
2	−25·19	194	+0·00289	+0·21	+0·13
3	−25·18	345	+ 218	+ ·07	+ ·01
4	−24·20	162	− 264	+ ·03	+ ·10
5	−23·21	210	− 241	− ·11	− ·04
6	−23·00	184	− 242	− ·02	+ ·05
7	−22·90	281	− 203	+ ·01	+ ·06
14	−19·73	344	+ 135	+ ·03	− ·01
15	−19·40	359	+ 129	+ ·07	+ ·04
16	−16·75	72	− 108	− ·02	+ ·01
18	−16·65	245	+ 117	+ ·03	·00
19	−16·64	326	−0·00098	·00	+0·03

A solution by least squares, giving unit weight to each observed magnitude, gives a correction of $+27''\pm 6''$ to q_c. The result from the lunar eclipse magnitudes is then

$$q = -23''\pm 6''. \qquad (20)$$

The final column, R, of Table I contains the residuals from this solution. The predominance of positive residuals suggests that the observed magnitudes may be systematically too large by about 0·03; this is in addition to an allowance of 0·02 for increment in the radius of the shadow due to the Earth's atmosphere, which has already been included in δG. The probable error of one magnitude is $\pm$ 0·04 and the magnitudes were in fact only recorded to the nearest digit (0·08). In view of the indefinite nature of the phenomenon the presence of a small

systematic error is hardly surprising. In any case its effect on the final result (20) is inappreciable.

8. The value of q derived from the ancient lunar eclipses (20) is in excellent agreement with that deduced from Hipparchus' observations (9). Combining these two independent determinations we may adopt

$$q = -21'' \pm 3'' \tag{21}$$

as the value which best fits the ancient observations discussed in this paper. This differs from the modern value (4) by nearly three times the sum of their probable errors. There is thus strong evidence that the lunar tidal couple may have changed considerably during the last twenty centuries. This is not improbable. The dissipation is thought to occur in relatively few regions of shallow sea; local changes in coastline and sea level in these regions will therefore have a large effect on the total dissipation.

The latest solution for the secular accelerations involving ancient observations is that by Brouwer (**10**); he, however, imposed the condition that q should have its modern value (4) throughout the whole period covered by his discussion, the validity of which procedure would now appear to be open to doubt.

Apart from changes in the mean longitude at epoch and the mean motion of the Moon, the revision of Brown's mean longitude adopted in constructing the *Improved Lunar Ephemeris* consists of the removal of one empirical term (the "great empirical term"), and the substitution of another (qT^2). This ephemeris is to be used as the standard of ephemeris time, all departures of the Moon's observed position from the ephemeris being attributed to deviations of the rotation of the Earth from uniformity. Any change in q will therefore introduce a systematic departure of E.T. from a truly uniform Newtonian time.

I wish to thank Dr R. d'E. Atkinson for his helpful criticism of this paper.

Royal Greenwich Observatory,
Herstmonceux Castle, Sussex:
1957 *June* 28.

References

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THE EFFECT OF BLANKETING ON THE STRUCTURE OF THE SOLAR ATMOSPHERE

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Summary

The effect of blanketing on the structure of the solar atmosphere has been investigated by computing a (physically impossible) model solar atmosphere with variable net flux of energy such as would result if the absorption in spectral lines were to cease while retaining the actual temperature distribution of the solar atmosphere. Pecker's (*Ann. d'Ap.*, **14**, 152, 1951) values of the fraction η absorbed in spectral lines have been taken as the basis for these investigations.

The investigations give a temperature drop above the level of mean optical depth $\tau = 0{\cdot}088$ and a rise of up to 225 degrees below this level.

1. *Introduction*.—In a previous paper, which subsequently will be referred to as (**1**), a model solar atmosphere with practically constant net flux of energy was computed. Thus the computed model satisfied the important condition of radiative equilibrium, but in spite of that the agreement between the observed limb darkening and the limb darkening computed theoretically from the model was poor. This disagreement between theory and observation is mainly due to the following three factors:

(i) the model did not take into consideration the effect of the absorption lines (blanketing effect), since only the absorption in the continuum was taken into account;

(ii) there may be a departure from radiative equilibrium due to convection in the unstable zone; and

(iii) the adopted values of the monochromatic absorption coefficients may be erroneous.

In the present paper it is intended to investigate the modification of the structure of the solar atmosphere due to the blanketing effect. An investigation of the other two sources of disagreement between theory and observation will be the subject of a subsequent paper.

2. *Previous investigations of the blanketing effect*.—In his classical paper on the blanketing effect, Milne (**2**) assumed that the spectral lines were formed in an optically thin outer layer of the solar atmosphere (reversing layer), and that they absorbed a certain fraction of the radiation and returned it to the photosphere. This reflected radiation could not accumulate indefinitely but escaped in the form of increased radiation in the continuum between the lines because of the increase of temperature in the photosphere under the reversing layer. The mathematical treatment of this problem gave an overall increase of the temperature in the solar atmosphere.

Chandrasekhar (**3**) assumed that the absorption lines were uniformly distributed in the whole spectral range and were formed throughout the whole atmosphere, avoiding in this way the artificial division of the solar atmosphere into a photosphere and a reversing layer. Assuming the ratio of the absorption coefficient in lines to the absorption coefficient in the continuum to be constant and independent of the depth and using Eddington's approximation, he solved the equation of transfer separately for the continuum and for the absorption lines under the assumption of a constant net flux of energy. His investigation showed that (i) true absorption lowered the temperature close to the boundary of the atmosphere and increased it in the deeper layers; and (ii) pure scattering did not change the boundary temperature but increased the temperature gradient in the stellar atmosphere. In addition, his treatment of Milne's problem of a reversing layer showed an increase of the temperature gradient in the reversing layer without any alteration in the gradient beneath it. The temperature below the reversing layer was, however, raised. All these results were obtained starting from the Milne-Eddington model of the solar atmosphere.

Hopf (**4**) supplemented the theory of Chandrasekhar by evaluating exactly the boundary temperature using the method applied in his tract (**5**) for the exact solution of the equation of transfer in a grey atmosphere. Münch (**6**) solved the problems dealt with by Chandrasekhar using the methods developed by Chandrasekhar (**7, 8, 9**) for the approximate evaluation of the source function by means of Gaussian sums. In addition, he generalized the solution of Chandrasekhar for the case of pure absorption assuming lines to be unevenly distributed throughout the spectrum and applied the results of his investigations to the solar atmosphere.

All the above attempts, with the exception of Münch's treatment, were of little practical value since the assumptions concerning the nature of absorption in lines and their distribution in the spectrum were too simplified and artificial. Pecker (**10**) and Labs (**11**) applied a method proposed by Strömgren and Unsöld (**12**) for correcting a model stellar atmosphere to models which probably approach closely to the real solar atmosphere. This method gives the corrected temperature at a given level from the assumption that the absorption in the continuum and spectral lines must be equal to the total emission

$$\int (\kappa_c + \kappa_\nu) J_\nu \, d\nu = \int (\kappa_c + \kappa_\nu) B_\nu \, d\nu. \tag{1}$$

Laborious computations were necessary in order to find the absorption coefficient κ_ν in spectral lines. It was, of course, not possible to deal with every line separately but good results were obtained where lines were grouped in classes according to their equivalent width, ionization potential, exitation potential and wave-length. In this way the absorption lines were classified into a number of line types and the work was greatly reduced. The method of improving model stellar atmospheres by means of the iterative equation (1) is, however, only slowly convergent and for this reason Labs (**11**), applying the formula only once, was not able to improve the temperature distribution below the mean optical depth $\tau = 0{\cdot}01$.

Pecker (**10**), assuming that formula (1) need be applied once only, and making other simplifying assumptions, derived the formula

$$\frac{\Delta T}{T}\left(1 - \frac{4T}{T}\right) = -\left\{\frac{d}{d\tau_{5000}}\left(\frac{\Delta F}{F}\right)\right\} \frac{\kappa_{5000}}{\int \kappa_\nu B_\nu \, d\nu / \int B_\nu \, d\nu} \left(\frac{T_e}{2T}\right)^4, \tag{2}$$

where ΔT is the required temperature correction, F the constant net flux of energy and ΔF the fraction of the flux absorbed in lines. Using this formula he calculated corrections down to the optical depth $\tau_{5000} = 0{\cdot}9$. However, since Labs, using the exact formula, was unable to obtain satisfactory corrections below $\tau = 0{\cdot}01$ it is extremely unlikely that Pecker's corrections based on an approximate formula are in any way reliable.

The disappointing results of Labs' investigations prompted Böhm (**13**) to apply the "flux-iterative" method of Unsöld (**14**) to the problem of blanketing. This method is based on the equation

$$-\pi \,.\, \Delta B = \tfrac{1}{2}\Delta F(0) + \tfrac{3}{4}\int_0^\tau \Delta F(\tau)\, d(\tau) - \tfrac{1}{4}\frac{d}{d\tau}\{\Delta F(\tau)\}, \tag{3}$$

in which ΔB is the required correction of the source function at the mean optical level τ and $\Delta F(\tau)$ is the difference between the actual net flux of energy and the required constant net flux.* Formula (3) has been derived by R. v. d. R. Woolley (**15**) for a grey atmosphere for which Eddington's approximation is valid. It gives only an approximate correction ΔB but a better value can be found from it with the aid of the exact iterative formula

$$\Delta B = \Lambda(\Delta B) - \frac{1}{4\pi}\frac{d}{d\tau}(\Delta F) \tag{4}$$

in which the Λ-operator is defined by the equation

$$\Lambda_x\{f(t)\} = \tfrac{1}{2}\int_x^\infty f(t)\, E_1(t-x)\, dt + \tfrac{1}{2}\int_0^x f(t)\, E_1(x-t)\, dt.$$

The solution of equation (4) must satisfy the boundary condition

$$\Delta F(0) = \pi\Phi_0(\Delta B)$$

where the Φ-operator is defined by the equation

$$\Phi_x\{f(t)\} = 2\int_x^\infty f(t)\, E_2(t-x)\, dt - 2\int_0^x f(t)\, E_2(x-t)\, dt.$$

The method of Unsöld gives good results since the iterative formula (4) is likely to be convergent for reasons given by Böhm in his paper. In the present paper, however, it is intended to apply another method to the blanketing problem and to compare the results with those of Böhm.

3. *Application of a variational method to the problem of the blanketing effect*

3.1. *Introductory remarks.*—The blanketing effect modifies the temperature distribution, which in consequence changes the continuum between the spectral lines. If, then, the absorption in lines were to cease while the modified temperature distribution were maintained, the total flux of energy in the solar atmosphere would not be constant but would vary with the depth. The actual value of the flux $F(\tau)$ at the level τ would be the observed constant flux $F_{\text{obs}}(0)$ increased by the amount of energy absorbed in spectral lines at this level τ. It is, therefore, possible to allow for the effect of blanketing on the structure of the star's atmosphere by computing a model of the solar atmosphere with variable total flux $F(\tau)$ dependent on the optical depth and disregarding the spectral lines altogether, i.e. by taking only the continuous absorption coefficient into account when computing the net flux of energy. This task of computing a model solar atmosphere with a variable

* In the present paper the symbol F for the net flux is as used by Woolley and Stibbs in *The Outer Layers of a Star* (Oxford, 1953). In the *Radiative Transfer* of Chandrasekhar (Oxford, 1950) and the *Basic Methods in Transfer Problems* of Kourganoff (Oxford, 1952) $\pi \,.\, F$ is used for this quantity.

flux of energy is *a priori* more difficult than the computation of a model with constant flux. However, the application of a variational method described by the author in (1) makes this task very easy if a model solar atmosphere with a constant or nearly constant flux of energy is available. Such a model has been published in Table V of (1): the net flux of energy (without taking the spectral lines into account) varies by 4·3 per cent from its boundary value $F^{(2)}(0) = 62{\cdot}68 \cdot 10^9$ ergs cm^{-2} sec^{-1} within the mean optical depth $\tau = 11{\cdot}528$. In the present paper it is intended to improve this model which subsequently will be referred to as the initial model. The designation $F^{(2)}(\tau)$ will be used for its net flux of energy in accordance with the notation used in (1). The values of $F^{(2)}(\tau)$ are shown in Table VI of (1).

3.2. *Theoretical remarks.*—In the variational method of improving stellar models described by the author in (1), the required corrections of the source function $\Delta B(\tau)$ are represented as a linear combination of some elementary functions $f_1, f_2, \ldots, f_n$:

$$\Delta B(\tau) = a_1 f_1(\tau) + a_2 f_2(\tau) + \ldots + a_n f_n(\tau). \tag{5}$$

The resulting change of the net flux is then

$$\Delta F(\tau) = a_1 \pi \Phi_\tau(f_1) + a_2 \pi \Phi_\tau(f_2) + \ldots + a_n \pi \Phi_\tau(f_n). \tag{6}$$

Assume now that the change of the net flux $\Delta F(\tau)$ is equal to the difference between the required (variable) flux $F(\tau)$ and the net flux $F^{(2)}(\tau)$ of the initial model. The system of equations (6) written for a sufficient number of optical levels τ can be solved for the unknowns $a_1, a_2, \ldots, a_n$, which, inserted in equation (5), give the correction of the source function $\Delta B(\tau)$ and consequently the correction of the temperature $\Delta T(\tau)$.

3.3. *Application to the solar model.*—An application of the above method in (1) was so successful that it could be applied with confidence to the present problem. The essential point of this method is that the same mean of the absorption coefficient is used in the computation of the mean optical depth for the initial and the corrected models. It can be expected that the temperature correction will have the simplest form if the true mean absorption coefficient

$$\kappa = \frac{1}{F}\int_0^\infty \kappa_\nu F_\nu \, d\nu$$

is used with the inclusion of the absorption in special lines. This, however, cannot be computed until the final model is known. An approximation of this mean can be computed for the continuous absorption coeffiicent with the formula

$$\kappa = \frac{1}{F^{(2)}(\tau)}\int_0^\infty \kappa_\nu F_\nu^{(2)}(\tau) \, d\nu \tag{7}$$

in which the monochromatic fluxes $F_\nu^{(2)}(\tau)$ are known from intermediate results of the computations in (1). The results of the computations of the mean (continuous) absorption coefficient κ with the formula (7) are shown in Table I. For reasons explained in (1) the values of the logarithm of the electron pressure p_e are given together with the values of κ.

According to Böhm (13) the absorption in spectral lines increases the mean absorption coefficient by about 10 per cent, and therefore the values of κ from Table I multiplied by 1·1 will be used for the computation of the mean optical depth, which will be designated τ^* for both the initial and the corrected (final) model. The notation τ will be used for the mean optical depth without taking the absorption in spectral lines into account. Since the optical depth of the initial model has been computed in terms of another mean absorption coefficient a re-computation of the

mean optical depth τ^* has been made for the purpose of the present paper. The results of this re-computation are shown in Table II where both the initial and the final model are shown side by side.

In order to avoid misunderstandings it should be stressed that the correction of the mean absorption coefficient due to spectral lines is used only for the computation of the reference depths in equations (5) and (6). (In the following sections these equations will be used with the notation τ^* for the optical depth.) If a check is made to ascertain whether the corrected model has the variable net flux of energy $F(\tau)$, as mentioned in section 3.1, only the continuous absorption coefficient should be taken into account. No check of that kind, however, will be made in the present paper, since the results of (**1**) show that the variational method of correcting model stellar atmospheres applied here is a reliable one.

TABLE I

Values of κ/p_e in units of 10^{-26} cm^4 dyne^{-1} per one heavy particle as defined by Vitense (**16**)

Θ	$\log_{10} p_e$	κ/p_e
1·04	0·149	9·989
1·00	0·599	7·524
0·90	1·074	4·936
0·80	1·647	3·164
0·70	2·372	2·113
0·60	3·137	1·547
0·50	3·898	1·434

3.4. *Fraction η of energy absorbed by spectral lines.*—The exact determination of the amount of energy absorbed by spectral lines is a very difficult task, since it is almost impossible to draw the continuous background in some parts of the spectrum. For this reason the evaluation of the fraction absorbed varies considerably from author to author. Thus for instance Mulders' determination (**17**) gave $\eta = 0{\cdot}083$, Michard's (**18**) $\eta = 0{\cdot}124$, Wempe's (**19**) $\eta = 0{\cdot}091$. Still less is known about the variation of this fraction η with the optical depth. Laborious computations have been undertaken by Pecker (**10**) and Labs (**11**) in order to determine this variation, but of course no great accuracy could be expected. Only the results of Pecker have been published and they will be taken as a basis for the present investigation. The uncertainty connected with that kind of investigation is so great that an independent re-computation would hardly be justified. This means, of course, that the modification of the structure of the solar atmosphere due to the blanketing effect can only be computed approximately.

3.5. *Numerical results.*—The following eight elementary functions

$$f_1(\tau^*) = 1, \qquad f_2(\tau^*) = \tau^*, \qquad f_3(\tau^*) = \tau^{*2}, \qquad f_4(\tau^*) = \tau^{*3},$$
$$f_5(\tau^*) = e^{-\tau}, \qquad f_6(\tau^*) = e^{-5\tau^*}, \qquad f_7(\tau^*) = e^{-10\tau^*}, \qquad f_8(\tau^*) = E_2(\tau^*)$$

have been used in order to find the corrections to the temperature distribution of the original model. The Φ-transforms of these functions are:

$$\Phi(f_1) = 2E_3(\tau^*), \qquad \Phi(f_2) = \tfrac{4}{3} - 2E_4(\tau^*), \qquad \Phi(f_3) = \tfrac{8}{3}\tau^* + 4E_5(\tau^*),$$
$$\Phi(f_4) = 4{\cdot}8 + 4\tau^{*2} - 12E_6(\tau^*), \quad \Phi(f_5) = 2\{e^{-\tau^*}(0{\cdot}72963 - \ln\tau^*) - E_1(\tau^*) - E_2(\tau^*)\},$$
$$\Phi(f_6) = 0{\cdot}08\, e^{-5\tau^*}\{9{\cdot}59453 + E_1(4\tau^*)\} - 0{\cdot}08\{5E_2(\tau^*) + E_1(\tau^*)\},$$
$$\Phi(f_7) = 0{\cdot}02\, e^{-10\tau^*}\{19{\cdot}79932 + E_1(9\tau^*)\} - 0{\cdot}02\{10E_2(\tau^*) + E_1(\tau^*)\}.$$

The numerical values of the Φ-transform of the function $f_8(\tau^*) = E_2(\tau^*)$ can be found in Kourganoff's handbook (**20**) up to $\tau^* = 3{\cdot}25$. For greater values of τ^* they were computed numerically by the author.

Equation (6) has been written for 21 levels of τ^* from 0 to 12·807. These levels were not uniformly distributed but there were more levels for small values of τ^* where the effect of blanketing is more important. Only seven levels were taken into account for $\tau^* \geqslant 3{\cdot}0$. It was intended to obtain a corrected model with a net flux of energy equal to the boundary value of the initial model $F^{(2)}(0) = 62{\cdot}678$ ergs cm^{-2} sec^{-1} corresponding to a solar constant 1·945 cal. cm^{-2} min^{-1}.

The solution of the system of equations (6) by the least-squares method gave the following results:

$$a_1 = +2{\cdot}432, \quad a_2 = +2{\cdot}837, \quad a_3 = -0{\cdot}348, \quad a_4 = +0{\cdot}0118,$$

$$a_5 = +13{\cdot}888, \quad a_6 = +10{\cdot}483, \quad a_7 = -11{\cdot}320, \quad a_8 = -16{\cdot}878.$$

With these coefficients equation (5) gives the corrections of the source function ΔB and thus the new temperature. For $\tau^* = 0$ the temperature is lowered by more than 1000 degrees; it remains unchanged for $\tau^* = 0{\cdot}088$, and then increases by an amount ranging up to 225 degrees. The sharp drop of the temperature close to the boundary is, however, less pronounced if fewer elementary functions f_n are used in equations (5) and (6), even though little is changed for levels τ^* over 0·2. Thus if only the first seven functions are used the boundary temperature is lowered by only 50 degrees; for the first six functions there is even a small increase of the boundary temperature. If only the first five functions are used, the solution gives an overall increase of the temperature by about 200 degrees. If, however, an additional function $f_9(\tau^*) = E_3(\tau^*)$ is used the system of equations (6) with nine parameters a_n is undetermined and this means that eight functions are sufficient to determine the temperature distribution of the corrected model solar atmosphere.

It must be stressed that the method used in this paper does not permit the determination of the temperature distribution in the uppermost parts of the solar atmosphere, where a sharp drop of the temperature can be expected. This is due to the fact that a large change of temperature in a sufficiently thin layer of the atmosphere produces only a negligible change of the net flux of energy. On the other hand the corrections in the deeper layers ($\tau^* \geqslant 0{\cdot}2$) are determined correctly.

Since the observational data (**21-25**) indicate a low boundary temperature of the Sun (down to about 3700 degrees) the temperature distribution obtained with eight functions f_n, lowering the boundary temperature to 3708 degrees, has been assumed for the computation of the corrected model solar atmosphere.

4. *Model solar atmosphere with blanketing effect.*—The method of computing corrected model stellar atmospheres has been described by the author in (**1**) and the same method has been used here. The results are given in Table II together with the initial model. For both models the optical depth τ is given in terms of the mean continuous absorption coefficient in the first column; a uniform increase of the absorption coefficient by 10 per cent is shown in the next column and allows approximately for the change of the mean absorption coefficient due to the absorption lines. The mean continuous absorption coefficient for the computation of τ has been taken from Table I; for the values of $\Theta = \dfrac{5040}{T} > 1{\cdot}04$ Rosseland's mean absorption coefficient has been used.

TABLE II

Model solar atmosphere

(p indicates total pressure due to hydrogen alone)

τ	τ^*	Initial model T	Initial model $\log_{10} p$	Initial model $\log_{10} p_e$	Corrected model T	Corrected model $\log_{10} p$	Corrected model $\log_{10} p_e$
0·00	0·00	4811°	$-\infty$	$-\infty$	3708°	$-\infty$	$-\infty$
0·01	0·011	4830	3·88	0·02	4076	3·91	$\bar{1}$·71
0·02	0·022	4849	4·04	0·16	4305	4·05	$\bar{1}$·98
0·03	0·033	4867	4·14	0·25	4459	4·13	0·12
0·04	0·044	4885	4·20	0·32	4611	4·21	0·22
0·05	0·055	4903	4·26	0·37	4723	4·26	0·31
0·06	0·066	4921	4·30	0·41	4814	4·30	0·38
0·07	0·077	4939	4·34	0·45	4891	4·34	0·40
0·08	0·088	4956	4·37	0·48	4958	4·37	0·42
0·09	0·099	4973	4·40	0·50	5018	4·40	0·52
0·10	0·110	4990	4·42	0·53	5070	4·42	0·55
0·12	0·132	5025	4·46	0·58	5154	4·47	0·62
0·14	0·154	5058	4·50	0·62	5222	4·50	0·67
0·16	0·176	5091	4·53	0·66	5279	4·54	0·72
0·18	0·198	5123	4·56	0·69	5327	4·56	0·76
0·20	0·220	5154	4·59	0·72	5368	4·59	0·80
0·25	0·275	5231	4·64	0·79	5455	4·64	0·87
0·30	0·330	5305	4·68	0·85	5529	4·68	0·94
0·35	0·385	5377	4·72	0·91	5595	4·72	1·00
0·40	0·440	5446	4·75	0·96	5656	4·75	1·05
0·45	0·495	5511	4·78	1·01	5714	4·78	1·10
0·5	0·550	5575	4·80	1·05	5771	4·80	1·15
0·6	0·660	5694	4·85	1·14	5881	4·84	1·25
0·7	0·770	5807	4·88	1·23	5986	4·86	1·34
0·8	0·880	5913	4·91	1·32	6085	4·89	1·43
0·9	0·990	6013	4·93	1·40	6180	4·91	1·51
1·0	1·10	6110	4·95	1·48	6271	4·93	1·59
1·2	1·32	6288	4·98	1·64	6442	4·95	1·74
1·4	1·54	6452	5·00	1·78	6600	4·97	1·88
1·6	1·76	6606	5·02	1·91	6748	4·99	2·00
1·8	1·98	6748	5·03	2·02	6886	5·00	2·11
2·0	2·20	6880	5·04	2·13	7016	5·01	2·21
2·5	2·75	7177	5·06	2·35	7309	5·03	2·43
3·0	3·30	7437	5·07	2·54	7566	5·04	2·61
3·5	3·85	7671	5·08	2·70	7798	5·05	2·76
4·0	4·40	7884	5·09	2·84	8008	5·06	2·89
4·5	4·95	8083	5·09	2·96	8202	5·06	3·01
5·0	5·50	8270	5·10	3·06	8381	5·07	3·11
6	6·60	8611	5·10	3·25	8708	5·07	3·28
7	7·70	8923	5·11	3·41	9002	5·08	3·43
8	8·80	9207	5·11	3·54	9270	5·08	3·55
9	9·90	9469	5·11	3·66	9518	5·08	3·66
10	11·0	9714	5·12	3·76	9750	5·08	3·75

As expected, the blanketing effect causes a sharp drop of the temperature in the outer layers of the solar atmosphere and a rise for layers below $\tau^* = 0{\cdot}088$. These results are shown graphically in Fig. 1, which can be considered as giving simple qualitative representation of the changes in structure of the solar atmosphere due to the blanketing effect.

5. *Comparison with other models*

5.1. *Boundary temperatures.*—The method used in the present paper does not provide any means of determining the effect of the blanketing on the outermost layers of the Sun. Thus the derived boundary temperature of 3708 degrees may be considerably out. Pecker (**10**) and Labs (**11**) in their investigations of the blanketing effect derived a boundary temperature of about 4300 degrees, while Böhm (**13**) found a temperature of 3400 degrees.

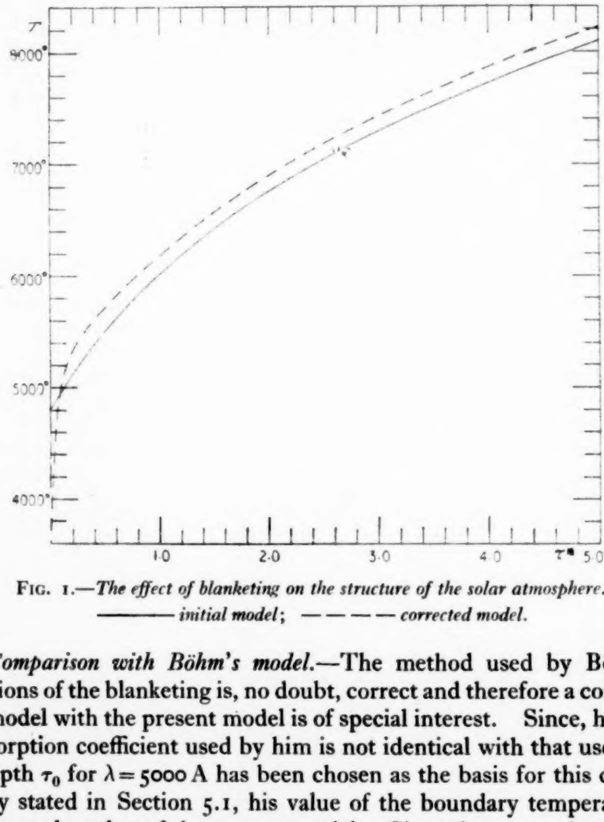

FIG. 1.—*The effect of blanketing on the structure of the solar atmosphere.* ——— *initial model*; — — — — *corrected model.*

5.2. *Comparison with Böhm's model.*—The method used by Böhm in his investigations of the blanketing is, no doubt, correct and therefore a comparison of his final model with the present model is of special interest. Since, however, the mean absorption coefficient used by him is not identical with that used here, the optical depth τ_0 for $\lambda = 5000$ A has been chosen as the basis for this comparison. As already stated in Section 5.1, his value of the boundary temperature is 300 degrees lower than that of the present model. Since, however, the temperature gradient of his model is much steeper, the temperatures of both models are equal (3840 degrees) at $\tau_0 = 0{\cdot}0007$. In the interval from $\tau_0 = 0{\cdot}0007$ to $\tau_0 = 0{\cdot}07$ the temperature of Böhm's model is up to 230 degrees higher while in the interval from $\tau_0 = 0{\cdot}07$ to $\tau_0 = 0{\cdot}6$ the difference is up to 130 degrees in the opposite direction. Below the level $\tau_0 = 0{\cdot}6$ Böhm's model shows a higher temperature and the difference reaches 105 degrees for $\tau_0 = 2{\cdot}0$. Böhm's model is not given for higher values of τ_0.

It is difficult to explain the differences between this model and that of Böhm. They may be due partly to the fact that Böhm's model was computed for a solar constant 1·96 cal cm^{-2} min^{-1}, while this model is computed for a slightly lower value (1·945 cal cm^{-2} min^{-1}). They may also be due to the use of a lower value

for the fraction of energy absorbed in spectral lines $\eta = 0{\cdot}109$. In addition, the variation of η with the optical depth is certainly not the same for both models.

Anticipating some results of further investigations on the model solar atmosphere, which will be published soon, it is possible to say that Böhm's model seems to represent the uppermost layer of the solar atmosphere better, i.e. where $\tau_0 < 0{\cdot}07$, while the present model may be better at greater depths.

5.3. *Comparison with empirical models.*—Fig. 2 shows a graphical comparison of the present model with the empirical model of Barbier (**26**) derived from the limb darkening of the continuum, that of de Jager (**27**) derived from the limb darkening of hydrogen lines and that of Vitense (**25**) derived from the investigation of metal lines. Differences of the order of 100 degrees exist between the four models represented on the diagram between $\tau_0 = 0{\cdot}0$ and $\tau_0 = 1{\cdot}0$. For $\tau_0 > 1{\cdot}0$ these differences are of the same order of magnitude but the empirical models are no longer reliable here.

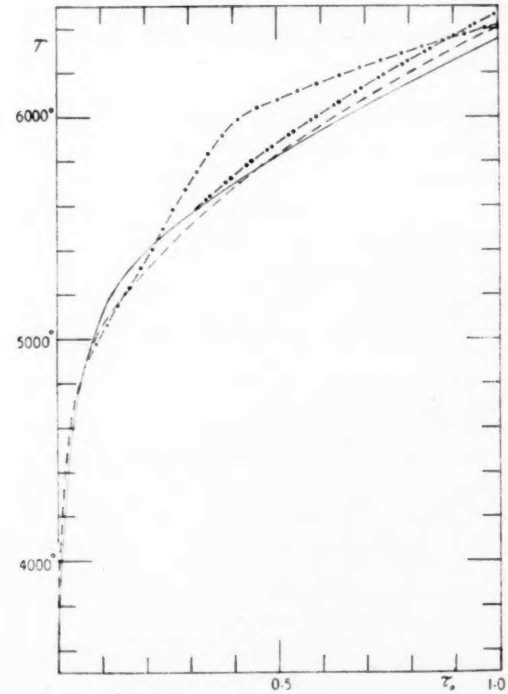

FIG. 2.—*Comparison of the theoretical model with the empirical models.*
— — — — — *Vitense's model I* —·—·—·— *de Jager's model VII*
—··—··—··— *Barbier's model* ———— *model under discussion*
(*In the interval from* 0·01 *to* 0·06 *de Jager's model is almost identical with that of Vitense.*)

6. ***Comparison with observational data.***—**A comparison of the limb darkening computed from the corrected model and the observed limb darkening shows considerable differences, even larger than for the initial model. This means**

that taking blanketing effect into consideration does not improve the model in this respect.

A further attempt to improve the model in order to remove the discrepancies between computation and observation will be described in a subsequent paper.

7. *Application of Unsöld's method.*—The application of formula (3) with the subsequent use of the iterative formula (4) should give the same corrections of the temperature as the variational method employed in this paper. Indeed, for $\tau^* > 0 \cdot 2$ the corrections of the source function computed with formula (3) do not differ by more than 8 per cent from those obtained in the present paper. It may be assumed that the subsequent application of the iterative formula (4) would remove these relatively small discrepancies, although no check has been made of this. For $\tau^* < 0 \cdot 2$ the differences are larger but there no determinations are reliable.

It may be concluded that Unsöld's method and the variational method used in the present paper give practically the same results.

8. *Acknowledgment.*—The present investigations were made on the suggestion of Dr R. v. d. R. Woolley, whose keen interest in the investigations by the author is gratefully acknowledged.

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TEMPERATURES AND ELECTRON DENSITIES IN FLARES AS DERIVED FROM SPECTROSCOPIC DATA

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Summary

It is shown that model flares with electron concentrations and temperatures in the ranges $5 \times 10^{11} - 10^{13}$ cm^{-3} and $10^4 - 1 \cdot 5 \times 10^4$ deg. K respectively have emissions consistent with the observed hydrogen and helium line intensities in solar flare spectra. Some difficulties are shown to exist in interpreting the observed great line width of Hα and a possible solution is given.

1. *Introduction.*—Over the past twenty years a good deal of observational data on solar flares has been obtained from studies both of their radiation at visible and radio frequencies and of their associated terrestrial effects such as SID's. Surveys of various aspects of present knowledge are given, e.g. by Ellison (**1**, **2**, **3**) and Kiepenheuer (**4**).

Data on spectra of flares are, however, rather meagre and show considerable variation from one observer to another and so, presumably, from flare to flare. Comparison of the spectra given by Allen (**5**), Ellison (**6**) and Richardson and Minkowski (**7**) shows that this variation is very marked in the helium lines $D_3(3^3D-2^3P)$ and $\lambda 6678 (3^1D-2^1P)$; the former appears, when at all, sometimes in emission and sometimes in absorption while the latter is seen only in emission from a few intense flares (**3**). Spectra covering the Balmer series limit, which will be of interest in the subsequent discussion, have been obtained by Richardson and Minowski (**7**) and by Suemoto (**8**).

The interpretation of flare observations in terms of a single model consistent with more than one aspect of the whole data does not seem to have been undertaken. The earliest attempt was by Giovanelli (**9**), who showed that, unless self-absorption of flare Lyman radiation were taken into account, temperatures of the order of 10^5 deg. K or more were required to account for the Hα intensity. In a more recent investigation Svestka (**10**) has interpreted the well known relationship between Hα central intensity and line width in terms of a model with temperature $\sim 1 \cdot 4 \times 10^4$ deg. K and hydrogen abundance N_H of the order of 10^{16} cm^{-3} ; Svestka's analysis, however, assumed thermodynamic equilibrium in a flare and his deductions are consequently open to some doubt. Suemoto (**8**), confining his attention to an analysis of the Balmer line H10, has concluded that if $T = 10^4$ deg. K, the electron concentration is $1 \cdot 1 \times 10^{12}$ cm^{-3}.

There is as yet no agreement as to the mechanism responsible for the wide Hα wings. Ellison and Hoyle (**11**) and Mustel and Severny (**12**) have suggested a Stark broadening mechanism. Goldberg, Dodson and Müller (**13**) employed a damping type of absorption coefficient in their analysis although they rightly pointed out that the main problem in connection with the Hα width is to account

for a sufficiently large number of II-state H atoms in the line of sight. Their estimate for this number, $\sim 10^{15}\,\mathrm{cm}^{-2}$, has been derived on the assumption of natural broadening but they have made no attempt to link this figure with flare thickness, temperature and hydrogen concentration.

Existing flare models thus seem to account for some observational features to the neglect of others. In attempting to gain fuller consistency with the main observed spectral features of flares, we are faced with considerable lack of knowledge of the structure of a flare in depth. Probably the most important factors in deciding the emission characteristics are the maximum electron concentration and the temperature in this region, together with a thickness ; this may be the actual thickness for a flare which is more or less uniform in height or a scale height if conditions vary through the flare.

We adopt as a flare model a layer, of uniform electron concentration N_e and in local statistical equilibrium at an electron temperature T, parallel to the colder solar photosphere which it overlies. The geometric form chosen is unimportant in the present context, for the computed emergent intensity will be only slightly affected by changes in shape. Absorption in the solar atmosphere above the flare is neglected ; measured flare heights indicate that generally there will be little absorbing material above this height in the chromosphere. Although we can hope to gain only an estimate of conditions in flares by using such a simple model, elaborations hardly seem justified in view of the paucity of observational data.

2. *Emission intensities in* Hα *and* D$_3$.—If the model flare layer is optically very thick at a particular frequency, so that no photospheric radiation penetrates the flare, the emergent intensity $I_\nu(0,\mu)$ in a direction making an angle $\cos^{-1}\mu$ with the outward normal can be found from the expression :

$$I_\nu(0,\mu)=\int_0^\infty \mathfrak{S}_\nu(\tau)e^{-\tau/\mu}\,d\tau/\mu \tag{1}$$

where $\mathfrak{S}_\nu(\tau)$ is a source function defined as the ratio of the monochromatic emission and extinction coefficients at the optical depth τ, i.e.,

$$\mathfrak{S}_\nu(\tau)=E_\nu/\kappa_\nu \tag{2}$$

For either coherent or noncoherent scattering in the medium, the emission coefficient E_ν at the line centre can be written in the approximate form,*

$$E_0=(1-\lambda)\frac{J_0\kappa_0}{4\pi}+\epsilon_0, \tag{3}$$

where λ is a scattering parameter defined so that $1-\lambda$ is the fraction of absorbed quanta which are subsequently re-emitted in the same line, ϵ_ν is a " true" emission coefficient, i.e., it represents the emitted component which is independent of absorption of radiation in the particular line, and J_ν is the total intensity defined by

$$J_\nu=\int_{4\pi} I_\nu(\mu)\,d\omega. \tag{4}$$

On the assumption that ϵ/κ and λ are constant throughout the flare, an expression for J_0 may be found using the Eddington approximation to the transfer equation

$$\tfrac{1}{3}\frac{d^2J}{d\tau^2}=\lambda J-4\pi\epsilon/\kappa, \tag{5}$$

* This follows, e.g., from the discussion given by Woolley and Stibbs (**14**) pp. 164 *et seq.*

where frequency subscripts have been omitted for convenience. Equation (5) has the solution, finite at large depths,

$$J = 4\pi\epsilon/\kappa\lambda + \alpha e^{-\sqrt{(3\lambda)}\tau}. \tag{6}$$

The integration constant α in (6) may be obtained as usual from the surface condition $J=(2/3)\, dJ/d\tau$ and it is then easily shown from (3), (2) and (1) that

$$I(0,\mu)=(\epsilon/\kappa\lambda)[1-(1-\lambda)/\{(1+2\sqrt{(\lambda/3)})(1+\mu\sqrt{(3\lambda)})\}]. \tag{7}$$

The quantities $\epsilon/\kappa\lambda$ and λ may be evaluated quite readily using methods given by Giovanelli and Jefferies (**15**) provided we know the various rates for collisional and radiative transitions between the atomic states. Collisional rates for any temperature may be found from appropriate cross section data and are given for H and He in the author's publications (**16**, **17**).* Radiative transitions are more troublesome since the intensity varies throughout the medium and is in any case only found by solving the equilibrium equations and the transfer equations. As pointed out in (**15**), a simplification is possible if the atmosphere is optically either thick or thin in the various spectral regions of interest. In the following the flare layer is assumed to be thick in the $L\alpha$ and $L\beta$ lines of H and in their counterparts in He, thin in the subsidiary continua, and thick for the principal (Lyman) continua unless otherwise specified. It can easily be shown from expressions for the optical thickness and ground state populations that, if the layer thickness is $\gtrsim 10^5$ cm and the helium abundance not improbably low, these conditions on the Lyman and Lyman type lines are met when $N_e \gtrsim 10^{12}\,\mathrm{cm}^{-3}$ and $T \lesssim 2{\cdot}5\times10^4$ deg. K ; anticipating the results of the present analysis it can be said that the derived conditions in flares conform to these limitations.

TABLE I

Intensity of the radiation emitted normally by a thick layer at the centre of Hα. *Surrounding continuum intensity at centre of disk is unity. Values marked with an asterisk are computed for a model optically thin in the Lyman continuum*

	T(deg. K)		
N_e(cm^{-3})	$1{\cdot}0\times10^4$	$1{\cdot}5\times10^4$	$2{\cdot}5\times10^4$
10^{11}	0·24	0·35*	0·48*
10^{12}	0·72	1·3	1·9*
10^{13}	1·8	2·8	4·0
10^{14}	2·6	4·4	5·4

TABLE II

Intensity of the radiation emitted normally by a thick layer at the centre of D_3. *Surrounding continuum intensity at the centre of the disk is unity*

	T(deg. K)			
N_e(cm^{-3})	$1{\cdot}0\times10^4$	$1{\cdot}25\times10^4$	$1{\cdot}5\times10^4$	$2{\cdot}5\times10^4$
10^{11}	0·28	0·34	0·39	0·50
10^{12}	0·99	1·1	1·7	1·9
10^{13}	2·4	2·8	3·4	4·3
10^{14}	3·6	4·0	5·4	6·3

Knowing $\epsilon/\kappa\lambda$ and λ, (7) may be evaluated and the emergent flare intensity expressed as a fraction of the appropriate continuum intensity at the centre of the Sun's disk (Münch (**19**)). Values obtained for the central intensities of Hα and D_3 in this way and so applying only to flares optically thick in these lines, in accordance with (1), are given as functions of N_e and T in Tables 1 and 2. These

* Some collisional rates given in (**16**) are too large by a factor of two. Corrections are given by Jefferies and Giovanelli (**18**).

figures should be reliable to 50 per cent or better ; at electron concentrations $\gtrsim 10^{12}$ cm^{-3} collisions generally predominate and errors in collision rates tend to cancel, due to corresponding errors in reverse transitions.

3. *Optical thicknesses in* H *and* He *spectra.*—The results given in Tables I and II apply to layers with high opacities at the line centres. Before making any comparison with observation, it is necessary to find expressions for the optical thicknesses of the layers in terms of their physical thicknesses z.

At the centres of Hα, D$_3$ and λ6678 the optical thicknesses are given, using a Doppler form absorption coefficient, by

$$\begin{aligned} \tau(\mathrm{H}\alpha) &= 4\cdot 9 \times 10^{-11} T^{-1/2} N_2(\mathrm{H}) z \\ \tau(\mathrm{D}_3) &= 6\cdot 3 \times 10^{-11} T^{-1/2} N_2{}^3(\mathrm{He}) z \\ \tau(6678) &= 8\cdot 6 \times 10^{-11} T^{-1/2} N_2{}^1(\mathrm{He}) z \end{aligned} \tag{8}$$

where it has been assumed that the hydrogen and helium fine structure states of a particular principal quantum number are distributed in the ratios of their statistical weights.

To evaluate the optical thicknesses from (8), values of the II-state populations are required. For He these are given in (**17**) and for H they can be found in a similar manner. Equation (8) then gives the results shown in Tables III, IV and V computed on the assumption of an H to He population abundance ratio of 5.

Table III

Values marked with an asterisk are computed for a model optically thin in Lc

$\tau(\mathrm{H}\alpha)/z$ (cm^{-1})

N_e(cm^{-3})	T(deg. K) $1\cdot0\times10^4$	$1\cdot5\times10^4$	$2\cdot5\times10^4$
10^{11}	$1\cdot2\times10^{-8}$	$1\cdot6\times10^{-8}$*	$6\cdot6\times10^{-9}$*
10^{12}	$4\cdot5\times10^{-7}$	$1\cdot2\times10^{-7}$	$1\cdot1\times10^{-7}$*
10^{13}	$7\cdot5\times10^{-6}$	$1\cdot4\times10^{-6}$	$4\cdot0\times10^{-7}$
10^{14}	$8\cdot0\times10^{-5}$	$1\cdot5\times10^{-5}$	$4\cdot2\times10^{-6}$

Table IV

$\tau(\mathrm{D}_3)/z$ (cm^{-1})

N_e(cm^{-3})	T(deg. K) $1\cdot0\times10^4$	$1\cdot25\times10^4$	$1\cdot5\times10^4$
10^{11}	$1\cdot2\times10^{-11}$	$1\cdot1\times10^{-9}$	$1\cdot1\times10^{-8}$
10^{12}	$1\cdot3\times10^{-10}$	$1\cdot6\times10^{-8}$	$2\cdot1\times10^{-7}$
10^{13}	$5\cdot0\times10^{-9}$	$1\cdot9\times10^{-7}$	$1\cdot1\times10^{-6}$
10^{14}	$1\cdot3\times10^{-8}$	$1\cdot9\times10^{-6}$	$2\cdot2\times10^{-5}$

Table V

$\tau(\lambda\,6678)/z$ (cm^{-1})

N_e(cm^{-3})	T(deg. K) $1\cdot0\times10^4$	$1\cdot25\times10^4$	$1\cdot5\times10^4$
10^{11}	$2\cdot7\times10^{-12}$	$2\cdot6\times10^{-10}$	$2\cdot7\times10^{-9}$
10^{12}	$2\cdot9\times10^{-11}$	$3\cdot8\times10^{-9}$	$6\cdot0\times10^{-8}$
10^{13}	$3\cdot4\times10^{-10}$	$5\cdot4\times10^{-8}$	$7\cdot0\times10^{-7}$
10^{14}	$3\cdot4\times10^{-9}$	$5\cdot7\times10^{-7}$	$7\cdot0\times10^{-6}$

4. *Comparison with observation.* (*a*) *Emitted intensities.*—The computed results given in the last two sections may now be compared with observational data. These are meagre and rather conflicting as far as simultaneous observations of Hα and D$_3$ are concerned. Thus Ellison (**1**) gives intensities for one flare only,

which in the above units are 3·0 for Hα and 1·2 for D_3. Richardson and Minkowski (**7**) recorded D_3 only for limb flares ; there is evidence, however, that these observations did not include any very strong flares. Allen (**5**) made visual estimates of the total changes in the Fraunhofer lines during a flare but his results do not lend themselves readily to the present discussion of central intensities.

Tables III and IV indicate that, particularly at the lower temperatures, the optical thickness in Hα will generally be considerably greater than in D_3. It may be noted that, since an underlying sunspot is not usually seen through a flare viewed at the centre of Hα, the flare is normally optically thick at this wavelength. There is no observational evidence for the D_3 optical thickness ; however, it is probably never very large since the line is observable only occasionally. Thus while the Hα central intensity generally corresponds to that for an optically thick model this is not the case for D_3, and so, while the central intensity of Hα should be as given in Table I, for D_3 the intensity should lie between unity—corresponding to radiation from the underlying photosphere—and the value given in Table II.

The closely similar values of the saturation intensities given in Tables I and II indicate that, if the D_3 line appears at all, Hα and D_3 should appear at the line centre either both in absorption or both in emission. The uncertainties in the computed data might allow for some departure from this expected behaviour, but it is interesting that Waldmeier (**20**) has found just this type of relationship between Hα and D_3, the transition from absorption to emission in D_3 occurring with an Hα central intensity of about 1·0.

It should also be pointed out that, with the noncoherent scattering mechanism probably applying for Hα, a double peaked profile will result—Jefferies (**21**)—as observed in most flares, and the maximum intensity across the line could exceed that at the centre by a considerable amount—a factor ~ 2 would not appear excessive. For D_3 this behaviour should not be so marked, since it will show a greater degree of coherence in scattering than Hα. Further, since the D_3 optical thickness is in general much smaller than Hα, the double peaks have less tendency to form.

The line λ 6678, which is the singlet counterpart of the triplet D_3, is only seen in the brightest flares (**1**) ; presumably the flare layer is optically very thin in this line for all but the strongest flares, and possibly even for them.

According to Richardson and Minkowski (**7**), and Suemoto (**8**), the flare emission at the Balmer series limit is small. If this is accepted as applying to most flares it may be used as a criterion for limiting their possible range of physical conditions. Thus the emission per unit solid angle from an atmosphere thin in the Balmer continuum is, at the series limit,

$$E_\nu = 2\cdot 1 \times 10^{-34} N_e^2 T^{-3/2} z \text{ ergs cm}^{-2}\text{ sec}^{-1}(\text{c/s})^{-1}$$

while the background radiation at the centre of the disk is, Münch (**19**),

$$I_\nu = 1\cdot 2 \times 10^{-5} \text{ ergs cm}^{-2}\text{ sec}^{-1}(\text{c/s})^{-1}$$

so that

$$E_\nu / I_\nu = 1\cdot 8 \times 10^{-29} N_e^2 z T^{-3/2}. \tag{9}$$

If this ratio is to be $< 5 \times 10^{-2}$ for Balmer continuum to escape observation, then

$$N_e^2 z T^{-3/2} < 3 \times 10^{27} \text{ cgs units.} \tag{10}$$

We now attempt to determine, using (10) and the results given in Tables I to V, the electron temperature and concentration for what might be considered a typical class 2–3 flare with low emission in $\lambda 6678$ and in the Balmer continuum, and an Hα emission at the line centre $\gtrsim 1 \cdot 0$ times the neighbouring continuum at the centre of the disk.

Fig. 1 shows the T, N_e space for the region $5 \times 10^3 \leqslant T \leqslant 2 \times 10^4$ deg. K, $10^{11} \leqslant N_e \leqslant 10^{14}$ cm^3, together with the zones prohibited to a model flare as giving theoretical results incompatible with observation. Two values of the model thickness, $z = 10^7$ and 10^8 cm, are used. The left hand curve, corresponding to the restriction on the Hα central intensity, is uncertain for $T < 10^4$ deg. K as theoretical results are lacking; this part of the curve is therefore shown by a broken line. The intensity of and opacity in D_3 do not materially limit the allowed T and N_e values and so for clarity its contributions to the figure are omitted.

The main virtue of the results given in Fig. 1 is that they show approximate limits to values of T and N_e; these values are subject to some uncertainty arising from errors in the basic data used in solving the equilibrium equations and also from possible error in the adopted helium abundance. For reasonable variations in these quantities, however, there will be only minor changes in the positions of the bounding curves.

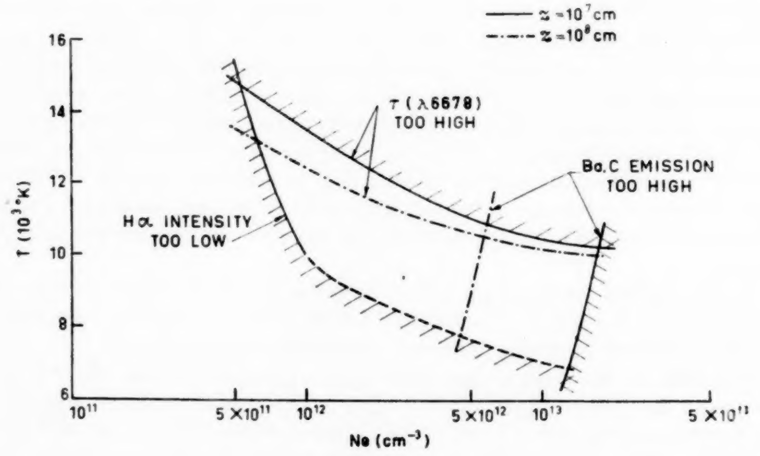

FIG. 1.

(*b*) *The* Hα *width*.—So far we have not attempted to explain the great width of the Hα line observed in solar flares, although clearly this most characteristic feature must be accounted for on any valid model. Observational data on the Hα line width are quite plentiful and have been investigated, e.g., by Goldberg *et al.* (**13**) and by Bruzek (**22**). These authors have shown that the Hα width depends strongly on the position of the flare on the disk, wider lines being nearer the limb, as might be expected. It appears from their analyses that we need to account for lines $\sim$4A wide at the disk centre. The line widths referred to have been measured visually and so refer to the difference between the two wavelengths for which the contrast between flare and background is just visible.

The estimated line width of a given flare thus depends on the contrast sensitivity of the observer, assumed here to be 2 per cent.

In the Hα wings, the flare intensity is given by an expression of the form,

$$I_\nu^{(f)} = I_\nu^{(p)}(1-\tau_\nu) + \int \epsilon_\nu \, dz \tag{11}$$

where $I_\nu^{(p)}$ is the monochromatic photospheric intensity and τ_ν is the optical thickness in Hα measured up from the photosphere and through the flare. We shall write $\tau = \tau^F + \tau^e$ where τ^F is the opacity of the flaring region and τ^e that of the material below the flare. Since Lyman radiation emitted by the flare will excite the underlying material, the contribution τ^e may be quite considerable.

The contrast P between the flare and its surroundings follows from (11) as

$$P_\nu = \frac{I_\nu^{(f)} - I_\nu^{(p)}}{I_\nu^{(p)}} = \frac{\int \epsilon_\nu \, dz}{I_\nu^{(p)}} - \tau_\nu, \tag{12}$$

which is readily shown to be equivalent to

$$P_\nu = \tau_\nu \left\{ \frac{4N_3 \exp(X_{\mathrm{H}\alpha}) - 1}{9N_2 f_\nu} \right\} \tag{13}$$

where (N_3/N_2) is an average population ratio taken over the emitting and absorbing region; $X_{\mathrm{H}\alpha} = h\nu/kT_r$, T_r being the radiation temperature (6150 deg. K), corresponding to the solar continuum near $H\alpha$, and $1 - f_\nu$, represents the fractional depression of the Hα wings below the continuum; at $\Delta\lambda = 2\mathrm{A}$, $f_\nu \simeq 0{\cdot}8$.

TABLE VI

Population of the hydrogen II-*state in* cm^{-3}. *The value marked with an asterisk is computed for a model optically thin in Lc*

N_e (cm^{-3})	T(deg. K) $1{\cdot}0 \times 10^4$	$1{\cdot}5 \times 10^4$
10^{11}	2×10^4	4×10^{4}*
10^{12}	9×10^5	3×10^5
10^{13}	$1{\cdot}5 \times 10^7$	4×10^6

For the flare models of interest here, the magnitude of the bracketed term in (13) is found to be only slightly dependent on N_e and to have values of about 2·5 and 5, respectively, for $T = 10^4$ and $1{\cdot}5 \times 10^4$ deg. K. Excitation by Lyman radiation will result in a similar value for this term in the region around a flare. To account for an Hα line of width 4A at the centre of the disk, we thus require, at $\Delta\lambda = 2\mathrm{A}$,

$$\tau = 8 \times 10^{-3}, \qquad (T = 1{\cdot}0 \times 10^4 \text{ deg. K})$$
$$\tau = 4 \times 10^{-3}, \qquad (T = 1{\cdot}5 \times 10^4 \text{ deg. K}), \tag{14}$$

To check whether these are compatible with the models described by Fig. 1, consider first the broadening mechanism. Which of the two most likely causes, statistical Stark effect or radiation damping, is dominant depends on the ion concentration; the former holds for $N_e \gtrsim 10^{13}\,\mathrm{cm}^{-3}$, the latter for lower N_e. The optical thicknesses follow from standard expressions as

$$\tau_s = 1{\cdot}5 \times 10^{-29} (\Delta\lambda)^{-5/2} \int N_e N_2 \, dz \tag{15}$$

for statistical Stark and

$$\tau_N = 5{\cdot}0 \times 10^{-17} (\Delta\lambda)^{-2} \int N_2 \, dz$$

for natural broadening, $\Delta\lambda$ being in Angstroms.

The component τ^F can be found using Table VI which gives, as a function of N_e and T, values of N_2 obtained by the solution of the equilibrium equations

for the II-state population of hydrogen. Table VII shows the resultant τ^F for two values of the model thickness and various electron temperatures and concentrations. Those models not compatible with Fig. 1 are indicated by an asterisk.

The other component, τ^c, of τ can be found as follows. Neglecting the Lyman emission of the underlying material and treating it simply as a purely absorbing and scattering medium, it is readily shown that the total intensity J' at the centre of $L\alpha$, say, below an optically thick flare is given by

$$J' = (2\pi\epsilon/\kappa\lambda)\exp[-\sqrt{(3\lambda)}\tau'],$$

where τ' is optical depth in the underlying medium and it is supposed that the scattering parameter λ has the same value as in the flare. The population of the II-state in the lower region follows from the usual equilibrium equation, which in the present case is simply

$$A_{12}N_1 = A_{21}N_2,$$

where $A_{12}N_1$ is the rate of absorption of $L\alpha$ radiation directed down from the flare. The number of II-state atoms in the line of sight is then given by

$$\int N_2\,dz = \int A_{12}N_1\,dz/A_{21}. \tag{16}$$

Supposing the $L\alpha$ line to be flat, of width $q\Delta\nu_D$, $\Delta\nu_D$ being the Doppler frequency width, $\int A_{12}N_1\,dz$ is then given by

$$\int A_{12}N_1\,dz = \frac{q\Delta\nu_D(2\pi\epsilon/\kappa\lambda)}{h\nu_0\sqrt{(3\lambda)}}. \tag{17}$$

Table VII

Values of the component τ^F ($\Delta\lambda$=2A) *for various flare conditions. The values marked with an asterisk correspond to models incompatible with Fig.* 1

T_e(deg K)	$z=10^7$cm, N_e (cm^{-3}) 10^{12}	$z=10^7$cm, 10^{13}	$z=10^8$cm, 10^{12}	$z=10^8$cm, 10^{13}
$1{\cdot}0\times10^4$	1×10^{-4}	4×10^{-3}	1×10^{-3}	4×10^{-2}*
$1{\cdot}5\times10^4$	4×10^{-5}	1×10^{-3}*	4×10^{-4}	1×10^{-2}*

For flare temperatures of 10^4 and $1{\cdot}5\times10^4$ deg. K, (17) has the approximate values $2\times10^{19}q$ and $2\times10^{21}q$ respectively, and these are only slightly dependent on the electron concentration of the flare. Taking $q=5$, corresponding to a flare of optical thickness 10^3 at the centre of $L\alpha$, we find, for $T=10^4$ deg. K,

$$\int N_2\,dz = 2\times10^{11}\,\text{cm}^{-2} \tag{18}$$

and for $T=1{\cdot}5\times10^4$ deg. K,

$$\int N_2\,dz = 2\times10^{13}\,\text{cm}^{-2}.$$

The resultant values of τ^c are given in Table VIII for natural and statistical Stark broadening. The electron concentration N'_e in this table is that of the region below the flare and has a value given by the equilibrium equation

$$\frac{Ne'}{N_1} \simeq \frac{P_{1c}}{P_{c1}+P_{c2}}, \tag{19}$$

the P's being transition rates to and from the continuum. The excitation rate P_{1c} is readily evaluated, knowing the Lc intensity, and is approximately $10^{-11}N_e^{(f)}$ or $10^{-9}N_e^{(f)}$ for flare temperatures of 10^4 or $1{\cdot}5\times10^4$ deg. K, respectively, $N_e^{(f)}$ being the flare electron concentration. The spontaneous rate $P_{c1}+P_{c2}$ is approximately $4\times10^{-13}N_e'$. At 500 km above the photosphere, $N_1\sim10^{14}\,\text{cm}^{-3}$

(23) while, at the photosphere, $N_1 \sim 10^{16}\,\text{cm}^{-3}$. If the Lc radiation penetrates to these levels, it follows from (19) that the ionization in the underlying region will be almost complete for a flare in which $10^4 \leqslant T \leqslant 1\cdot5 \times 10^4$ deg. K and $10^{12} \leqslant N_e \leqslant 10^{13}\,\text{cm}^{-3}$.

From Tables VII and VIII and the above discussion, it appears that in general for flare temperatures $\sim 1\cdot5 \times 10^4$ deg. K, we should expect $\tau^c > \tau^F$, while at lower temperatures $\tau^F > \tau^c$. In either case, it seems possible to find a flare model which satisfies the requirement (14) and is compatible with Fig. 1. However, consideration of the observed relationship between Hα width and central intensity and the variation of this width with time indicates that the τ^c component predominates at least in the initial stages of large flares, for the following reasons. At temperatures $\geqslant 10^4$ deg. K, hydrogen is largely ionized and, as shown in Table VI, a decrease in temperature increases N_2 in the flare and so increases τ^F. While the bracketed term in (13) decreases with decreasing temperature, the net result is that, if τ^F were the principal component of τ, an increase—or at best a very slow decrease—of line width would accompany a reduction in temperature and so in central intensity. This is in disagreement with observation, and it is especially difficult to see how the observed rapid initial decrease in line width from the maximum value could be accounted for by Hα absorption and emission processes in the flare itself.

Table VIII

Values of the component $\tau^c(\Delta\lambda = 2\text{A})$ for two flare temperatures

T_e(deg. K)	$\tau_N^{\,c}$	$\tau_S^{\,c}$
$1\cdot0 \times 10^4$	2×10^{-6}	$6 \times 10^{-19} N_e'$
$1\cdot5 \times 10^4$	2×10^{-4}	$6 \times 10^{-17} N_e'$

The extension by Lyman radiation of the flare excitation conditions into the surrounding material can, however, explain these emission characteristics in the Hα wings. As a flare cools from an electron temperature of $1\cdot5 \times 10^4$ deg. K, the Lyman radiation intensities decrease rapidly and so, according to these arguments, does the Hα line width. As the cooling progresses, excitation in the surroundings becomes too small to be of significance, and the Hα width is then controlled by excitation in the flare itself. It appears, then, that at the maximum phase of a large flare, the Hα line is Stark-broadened and, as the cooling progresses, natural broadening takes over since, from Fig. 1, an electron concentration $< 10^{13}\,\text{cm}^{-3}$ is suggested. In this later phase, where the line width is controlled by Hα emission and absorption processes within the flaring region itself, one would expect only slow changes of line width with changing Hα intensity.

(*c*) *The Lyman spectrum and ionospheric disturbances.*—The ionospheric effects of solar flares have often been ascribed to enhancement of emission in the Lyman series and continuum and it is thus of interest to compare the computed emission with estimates of the normal solar background in these frequencies.

The total intensity J_ν of the emergent radiation from a scattering and emitting medium of high opacity is given by the approximate relation :

$$J_\nu = \frac{2\sqrt{(\lambda/3)}}{1 + 2\sqrt{(\lambda/3)}} \cdot \frac{4\pi\epsilon}{\kappa\lambda}. \tag{20}$$

For $L\alpha$ and Lc, (20) may be readily evaluated by the methods indicated in (15). Resultant values of J_ν at the centre of $L\alpha$ and at the head of the Lyman continuum are given in Tables IX and X.

From an assessment of results of rocket observations, Watanabe *et al.* (**24**) have concluded that the $L\alpha$ flux at the earth is $\sim 2 \times 10^{-1}$ ergs cm^{-2} sec^{-1}. Byram *et al.* (**25**) have given more recent values which indicate a considerable temporal variation of this flux, figures as high as 9 ergs cm^{-2} sec^{-1} having been recorded in recent rocket flights. Taking the earlier figures as a lower limit and adopting a line width of 0·2A (**26**), the equivalent black body temperature of the sun at $L\alpha$ is found to be $6{\cdot}6 \times 10^3$ deg. K and $J \sim 2 \times 10^{-8}$ c.g.s. units. For Lc no observations have been made, but for an effective black body temperature of $6{\cdot}0 \times 10^3$ deg. K, $J \sim 1 \times 10^{-11}$ c.g.s. units.

Unpublished results of Gardner (**27**) on ionospheric observations at times of large solar flares show that the electron density at about 70 km can increase by a factor ~ 20. Since the equilibrium value of the electron density varies as the square root of the intensity of the ionizing radiations, an increase of $\sim 4 \times 10^2$ in the incident intensity is required if the same radiation is involved in causing normal and disturbed ionization. Since a large flare has area $\sim 10^{-3}$ of the visible hemisphere, the intensity of the incident radiation must then increase by a factor $\sim 4 \times 10^5$ at the flare. Comparison of Table IX with the background $L\alpha$ flux shows that such a large increase is not obtainable with models compatible with Fig. 1 unless the line width increases to an extreme value. For Lc, results cannot be definite since the background is unknown ; it appears from Table X that, if this background corresponds to that from a black body at 6×10^3 deg. K, the Lc flux from a flare may be some 10^5 times the normal background.

Table IX

Emitted intensity in c.g.s. units at the centre of $L\alpha$ for optically thick layers

	T(deg. K)	
N_e(cm^{-3})	$1{\cdot}0 \times 10^4$	$1{\cdot}5 \times 10^4$
10^{12}	$1{\cdot}2 \times 10^{-7}$	$5{\cdot}9 \times 10^{-6}$
10^{13}	$3{\cdot}4 \times 10^{-7}$	$1{\cdot}9 \times 10^{-5}$

Table X

Emitted intensity in c.g.s. units at the head of the Lyman continuum for optically thick layers

	T(deg. K)	
N_e(cm^{-3})	$1{\cdot}0 \times 10^4$	$1{\cdot}5 \times 10^4$
10^{12}	$1{\cdot}3 \times 10^{-8}$	$9{\cdot}7 \times 10^{-7}$
10^{13}	$8{\cdot}7 \times 10^{-8}$	$7{\cdot}8 \times 10^{-6}$

It seems from the above that we cannot ascribe both normal and enhanced D layer ionization to the absorption of $L\alpha$ nor by inference to any other of the Lyman lines. There is insufficient observational evidence to make any similar conclusion for Lc radiation.

A possible alternative source of ionizing radiation might be the helium resonance lines and continua which, because of their higher excitation potentials, will be enhanced to an even greater degree than the corresponding hydrogen emission. There is, however, as shown by C. Warwick (**28**), an additional condition which the ionizing radiation must satisfy. Her analysis of the association between heights of limb flares and the occurrence of SID's suggested that the ionizing radiation responsible for the production of an SID is absorbed in tangential passage through the solar atmosphere unless formed at a height above about 15 000 km. From an investigation of the opacity of the chromosphere in various possible ionizing radiations—X-rays, Lc, $L\beta$, and $L\alpha$—Warwick concluded from the above criterion that $L\alpha$ is the most probable cause of the SID's.

Since, however, the first helium resonance line at λ 584 has an absorption coefficient almost equal to that of $L\alpha$ and since, also, the number of ground state He atoms at 15 000 km above the solar surface is at least as large as for hydrogen, the tangential opacity in λ 584 is at least as large as that of $L\alpha$. Warwick's observations, therefore, could be interpreted equally well in terms of the helium resonance line as the SID producing agent.

There are, however, difficulties associated with this interpretation. Firstly, the actual flux in the helium resonance lines is low even for a flare at 15 000 deg. K, although the flux increases rapidly with increasing temperature and may be sufficient to produce SID's with a large flare. Furthermore, published data on absorption cross-sections and atmospheric concentrations suggest that these resonance lines will be absorbed at $\sim$180 km above the Earth's surface. The usual statement that the radiation cannot penetrate to the 70 km region is, however, not necessarily correct; the radiation is not all lost on absorption since, as the atom density is low, a good deal will be re-radiated by spontaneous recombination so that the radiation is effectively scattered and may penetrate to much lower levels than that at which it was first absorbed. However, excess ionization would be produced from the 180 km level down, and the fact that this is not observed argues against attributing SID's to radiation, such as the helium resonance lines, which is absorbed first at high levels in the atmosphere.

5. *Conclusions.*—Generally well verified observational results on the hydrogen and helium emissions of class 2–3 flares have been shown to be consistent with a uniform and static flare model whose electron temperatures and concentrations lie respectively in the ranges $1\cdot0 \times 10^4$ to $1\cdot5 \times 10^4$ deg. K and 5×10^{11} to $\sim 10^{13}$ cm^{-3}.

It has been shown that excitation in lower regions due to Lyman radiation from the flare is adequate to explain the $H\alpha$ line width for a flare temperature $\sim 1\cdot5 \times 10^4$ deg. K, the line being Stark-broadened at these levels. On this basis the observed relationship between Hα central intensity and the line width may be accounted for, and the observed rapid decrease of line width away from the maximum receives a ready explanation. It has also been shown that the $L\alpha$ flux flare is insufficient to account for the increased ionization observed simultaneously in the D layer, and that there are difficulties in attributing it to the helium resonance line λ 584.

For a complete analysis along the lines indicated in this work far more data are required on flare spectra. In particular, simultaneous measurements of the helium and hydrogen spectra for both limb and disk flares are urgently needed and observations in the region of the Balmer continuum could furnish very useful data for limiting the possible range of physical conditions in flares.

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Notes added in Proof

1. The referee has drawn the author's attention to two interesting recent publications by Svestka* on the physical conditions in flares ; the results of these are generally in agreement with those found above. It is interesting to note that he also has been forced to the conclusion that Lyman α radiation into the layers below the flaring region must be considered to explain the Hα width. He does not, however, consider the influence of Lyman continuum in ionizing this material and so producing Stark broadening. Further if the Hα opacity arises largely from a region below the flare, then Svestka's analysis based on the assumption that the $n=10$ state atoms—whose total number is derived from the Hα width—are distributed uniformly through the whole flare cannot be valid. Although, in general, Svestka recognizes the importance of departures from thermodynamic equilibrium, he implies in places an equality between kinetic and excitation temperatures. This is not acceptable and the run of temperature with depth derived this way is invalid since the excitation temperature must have an intrinsic increase with depth due to the increase in radiation intensity into the flare.

2. A recently completed investigation by the author on the influence of non-coherent scattering on the shape of emission lines indicates that the emergent Lyman α line from a flare will be strongly self reversed and that, while the central intensity given here should be correct, in the wings the intensity may increase by a factor $\sim 10^2$. Even so it seems difficult to account for the observed effects in the D region in terms of excitation by Lyman α.

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POSITIONS OF COMET AREND–ROLAND (1956 h)

R. H. Garstang

(Communicated by the Director of the University of London Observatory)

(Received 1957 June 22)

The following positions of Comet Arend–Roland (1956 h) have been obtained with the Radcliffe 24-inch photographic refractor at the University of London Observatory.

	1957	U.T.	1950·0 α	1950·0 δ	Parallax Factor α	Parallax Factor δ
			h m s	° ′ ″	s	″
1	Apr.	24·87249	2 33 32·26	+43 12 12·4	+0·394	+7·31
2	Apr.	27·89564	3 11 26·74	+51 9 28·5	+0·443	+7·04
3	Apr.	27·91900	3 11 45·11	+51 12 30·2	+0·387	+7·47
4	Apr.	28·87160	3 24 16·63	+53 9 35·1	+0·532	+6·24
5	Apr.	29·89831	3 37 56·04	+54 59 39·1	+0·513	+6·59
6	Apr.	30·91324	3 51 28·65	+56 33 25·2	+0·513	+6·67
7	May	13·89651	6 16 26·12	+63 37 22·3	+0·809	+3·88
8	May	13·90794	6 16 31·79	+63 37 23·5	+0·797	+4·22
9	May	22·92768	7 16 40·00	+63 16 41·0	+0·785	+4·34
10	May	22·94846	7 16 46·67	+63 16 34·4	+0·756	+4·95

The comparison star places were taken from the *Zweiter Katalog der Astronomischen Gesellschaft* (AGK 2), proper motions not being included.

Comparison Stars

1	+43° 296,	+43° 299,	+42° 271
2, 3	+51° 303,	+51° 308,	+50° 351
4	+53° 316,	+53° 320,	+52° 349
5	+55° 352,	+55° 358,	+54° 372
6	+57° 436,	+56° 437,	+56° 434
7, 8	+64° 346,	+63° 410,	+63° 409
9, 10	+63° 445,	+63° 450,	+63° 448

I am indebted to Mr T. Kiang for making his plate of April 24 available to me for measurement, and to Lieut Cdr L. M. Dougherty for voluntary assistance on several evenings.

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IMPERFECTIONS OF ELASTICITY IN THE SMALL BODIES OF THE SOLAR SYSTEM

Harold Jeffreys

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Summary

The rough estimate of the damping of the 14-monthly variation of latitude gives two types of solution on the hypothesis of elastic afterworking, in which the times needed to attain the strain for long-continued stress are of the orders of a few weeks and nearly a year. These are applied to tidal friction in the Moon and Mercury. The long scale leads to quantitative difficulties. For the Moon the straightforward application, in which it is supposed that the imperfection of elasticity is elastic afterworking and applies to the whole body, gives satisfactory results, but if it applies only to the inner part it is apparently necessary that the rotation and revolution periods were brought into close coincidence when the Moon was much nearer the Earth than it is now. For Mercury the effect is too small to have produced the required result in the time available, and a greater degree of imperfection of elasticity seems to be required, at least for periods of the order of a day or at some earlier stage of its history. The shorter time scale for the relaxation leads to no serious difficulties.

Comparison with satellites of the outer planets shows some peculiar features, but these may be attributed to differences of composition.

1. It is well known that many bodies in the Solar System keep a constant face towards their primaries, and that tidal friction provides a qualitative explanation ; further, no other explanation has been suggested. The work of Darwin was mostly concerned with tidal friction in the body of the Earth, treated as a highly viscous liquid; but there was no independent evidence of the value of the viscosity. Later work has shown that bodily imperfections of elasticity in the Earth cannot account for the apparent secular accelerations of the Moon and Sun, which must be attributed to turbulence in seas with fast currents. Most of the bodies concerned have no such seas. Positive evidence for any imperfection of elasticity of the Earth under small stresses has been hard to find, but some has at last appeared from the damping of the 14-monthly variation of latitude. It is therefore desirable to examine whether this degree of imperfection of elasticity is adequate to account for the rotations of the Moon and Mercury and possibly other satellites. In all cases, besides the steady rotation and revolution, there are theoretically forced and free vibrations. Some of the forced ones are perceptible for the Moon ; the free ones are too small to have been detected. For other bodies even rather large librations could not be detected. A further problem for the Moon is to explain how the free librations, probably originally large, could have been damped enough to be imperceptible.

In an early paper (Jeffreys 1915), I discussed changes of figure of the Earth and Moon under the influence of changes of rotation due to tidal friction, and

the possibility that the Moon's excess ellipticities are the result of its having solidified when much nearer the Earth than it is now. I found that even if the Moon has behaved as if perfectly elastic the smallness of the free libration in longitude is consistent with this hypothesis, provided that the periods of rotation and orbital motion had been brought into coincidence before solidification. Even if there was considerable libration at solidification it has had ample time to die down if the Moon has the viscosity of pitch, which was taken as of order 10^{10} c.g.s., but there were difficulties about the persistence of the ellipticities on the hypothesis of finite viscosity, and in any case this viscosity is far too small.

2. *Libration.*—In the first place I consider the elastic case, since the equations for this case can be easily adapted to various types of linear imperfection of elasticity. The analysis is similar to that of Jeffreys and Vicente (1957a, b, referred to as I and II) for the Earth's nutations. The displacements were expressed by small rotations as for a rigid body, with an elastic set superposed that makes no contribution to the angular momentum. The elastic displacements correspond to co-ordinates that would have short periods in a free vibration, and can be eliminated at an early stage. Additional displacements had to be introduced to represent the motion of a liquid core, but these need not be considered here. The main other differences are as follows. (1) A full treatment of the elastic displacements of a planet must take account of initial stress. Love (1911) introduced the hypothesis that the initial stress is hydrostatic pressure, and we adopted this for the Earth, for which it is a good approximation. The librations of the Moon, however, depend on the ellipticities, which far exceed the hydrostatic values, and it is desirable to examine directly whether the associated initial stress-differences have any importance. (2) The usual theory treats the Moon as a rigid body, and the initial stresses do not matter. But elasticity must be considered before we can consider damping by imperfections of elasticity. There are theoretically three free periods of libration, none of which has been established by observation. Their ratios are of the order of 60, and for some types of imperfection of elasticity they might be very differently damped. The fact that they seem to have disappeared, combined with the persistence of the ellipticities themselves, should give useful inequalities for the imperfections in question. (3) For the Earth a term representing the changes of obliquity has to be added to the gravitational potential in determining the elastic displacements. For the Moon there is another term due to the potential produced by the Earth. (4) For the Earth the angular velocity about the axis of greatest moment can be treated as constant, but for the Moon extra displacements in longitude are an essential part of the problem, and a new co-ordinate is needed to represent them.

2.1. Let the initial stress be $-p_0\delta_{ik}+q_{ik}$, with $q_{ii}=0$. The undisturbed geopotential is Ψ. The equations of equilibrium are

$$\rho_0\frac{\partial\Psi}{\partial x_i}-\frac{\partial p_0}{\partial x_i}+\frac{\partial q_{ik}}{\partial x_k}=0. \tag{1}$$

The surface stress in the steady state is

$$-l_i p_0+l_k q_{ik}, \tag{2}$$

and is zero since the surface is free. Taking the work done by the stresses q_{ik} to the first order, we must replace W_p of I by

$$W_p = \iint p_0\left(l_i u_i + \tfrac{1}{2} l_i u_i \frac{\partial u_k}{\partial x_k} - \tfrac{1}{2} l_k u_i \frac{\partial u_k}{\partial x_i}\right) dS$$
$$- \iiint u_i \left\{\frac{\partial p_0}{\partial x_i} + \tfrac{1}{2} u_i \frac{\partial}{\partial x_i}\left(p_0 \frac{\partial u_k}{\partial x_k}\right) - \tfrac{1}{2} u_i \frac{\partial}{\partial x_k}\left(p_0 \frac{\partial u_k}{\partial x_i}\right)\right\} d\tau$$
$$- \iiint q_{ik} \frac{\partial u_k}{\partial x_i} d\tau. \tag{3}$$

The last term is

$$- \iint l_i q_{ik} u_k dS + \iiint u_k \frac{\partial q_{ik}}{\partial x_i} d\tau. \tag{4}$$

The surface integrals cancel to the first order by (2) and the volume integral in W_g arising from Ψ is, to the first order,

$$\iiint \rho_0 u_i \frac{\partial \Psi}{\partial x_i} d\tau.$$

The contributions from p_0, q_{ik} and Ψ cancel by the equations of equilibrium. Thus, as we should expect, initial stress introduces no new first-order terms. The second order terms are of the form $q_{ik} e_{im} e_{km}$; but the elasticity terms are of the same form with the elastic moduli replacing q_{ik}. Hence the second order terms arising from initial stress-differences would only be of the order of 1/1,000 of the elastic terms even if the material was on the verge of fracture. They are therefore negligible and the previous form of $W_g + W_p$ holds.

2.2. As for the Earth we take the axis of x_3 in the mean direction of the minor axis and the other axes rotating about it with constant angular velocity n. Rotations l, m of x_3 towards x_1 and x_2 are introduced; we take also an extra displacement χ about x_3. A particle at (x_1, x_2, x_3) goes to (ξ_1, ξ_2, ξ_3), where, by 2.2 of I,

$$\xi_1 = x_1(1 - \tfrac{1}{2}l^2 - \tfrac{1}{2}\chi^2) + u_1' + l(x_3 + u_3') - (\chi + \tfrac{1}{2}lm)(x_2 + u_2'),$$
$$\xi_2 = x_2(1 - \tfrac{1}{2}m^2 - \tfrac{1}{2}\chi^2) + u_2' + m(x_3 + u_3') - (\chi - \tfrac{1}{2}lm)(x_1 + u_1'),$$
$$\xi_3 = x_3(1 - \tfrac{1}{2}l^2 - \tfrac{1}{2}m^2) + u_3' - (l + m\chi)(x_1 + u_1') - (m - l\chi)(x_2 + u_2'). \tag{5}$$

2.3. If the standard position of the Earth is on the axis of x_1 the extra terms in Ψ are

$$\tfrac{1}{2}n^2(x_1^2 + x_2^2) + \frac{n^2}{2(1+\kappa)}(2x_1^2 - x_2^2 - x_3^2), \tag{6}$$

where κ is the ratio of the masses. Their contribution to W is

$$\tfrac{1}{2}n^2\left[l^2(C-A) + m^2(C-B) - \iiint 2\rho_0 u_i' \frac{\partial}{\partial x_i}\{x_3(lx_1 + mx_2)\} d\tau\right]$$
$$- \frac{n^2}{2(1+\kappa)}\left[3l^2(C-A) + 3\chi^2(B-A) - 6\iiint \rho_0 u_i' \frac{\partial}{\partial x_i}\{x_1(lx_3 - \chi x_2)\} d\tau\right] \tag{7}$$

Since we shall consider only free vibrations we need not consider displacements of the Earth from the axis of x_1.

Apart from a perfect differential,

$$n\iiint \rho_0(\xi_1\dot{\xi}_2 - \dot{\xi}_1\xi_2) d\tau = \tfrac{1}{2}n(B + A - C)(l\dot{m} - \dot{l}m)$$
$$+ n\iiint \rho_0 u_i' \frac{\partial}{\partial x_i}(-\dot{l}x_2x_3 + \dot{m}x_1x_3 + 2\dot{\chi}x_1x_2) d\tau, \tag{8}$$

$$\tfrac{1}{2}\iiint \rho_0(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2) d\tau = \tfrac{1}{2}(B\dot{l}^2 + A\dot{m}^2 + C\dot{\chi}^2). \tag{9}$$

We have used the conditions that u_i' are chosen to give no angular momentum and that the associated speeds of free vibration are high. Their contributions to W are equivalent to introducing an extra gravitation potential

$$U_1' = n^2 x_3(lx_1 + mx_2) + \frac{3n^2}{(1+\kappa)}(lx_1x_3 - \chi x_1 x_2)$$
$$+ n(-\dot{l}x_2x_3 + \dot{m}x_1x_3 + 2\dot{\chi}x_1x_2). \tag{10}$$

and the Lagrangian function is

$$L = \tfrac{1}{2}(B\dot{l}^2 + A\dot{m}^2 + C\dot{\chi}^2) + \tfrac{1}{2}n(B+A-C)(l\dot{m} - \dot{l}m) - \tfrac{1}{2}n^2\{l^2(C-A) + m^2(C-B)\}$$
$$- \frac{n^2}{2(1+\kappa)}\{3l^2(C-A) + 3\chi^2(B-A)\}$$
$$+ \iiint \rho_0 u_i'\left(\frac{\partial U_1'}{\partial x_i} + \frac{1}{2}\frac{\partial U_2'}{\partial x_i}\right)d\tau - \frac{1}{2}\iiint (\lambda\Delta'\delta_{ik} + 2\mu e'_{ik})\frac{\partial u_i'}{\partial x_k}\,d\tau$$
$$+ \iiint \rho_0 u_i'\left(\frac{1}{2}u_k'\frac{\partial^2\Psi}{\partial x_i \partial x_k} - \frac{1}{2}\frac{\partial \Psi}{\partial x_i}\frac{\partial u_k'}{\partial x_k} + \frac{1}{2}\frac{\partial\Psi}{\partial x_k}\frac{\partial u_k'}{\partial x_i}\right)d\tau. \tag{11}$$

u_i' are quasi-statical co-ordinates, and if a statical solution is used for them all quadratic terms in them disappear and the linear term (namely that in U_1') is halved. Then the terms in u_i' can be replaced by

$$\frac{1}{2}\iiint \rho_0 u_i'\frac{\partial U_1'}{\partial x_i}\,d\tau = \frac{1}{2}\iint \rho_0 l_i u_i' U_1' dS - \frac{1}{2}\iiint U_1'\frac{\partial}{\partial x_i}(\rho_0 u_i')d\tau. \tag{12}$$

It is enough to treat the Moon as homogeneous and incompressible ; then the volume integral vanishes. For $U_1' = k_2K_2$, with K_2 any solid harmonic of degree 2, the normal displacement is

$$l_i u_i = \epsilon K_2/a, \tag{13}$$

where

$$\epsilon = \frac{5k_2\rho a^2}{19\mu + 2g\rho a} \tag{14}$$

In the case

$$k_2K_2 = \tfrac{1}{2}n^2r^2(\tfrac{1}{3} - \cos^2\theta), \tag{15}$$

corresponding to the effect of a steady rotation, we get

$$\epsilon = \frac{5n^2\rho a^2}{2(19\mu + 2g\rho a)} = \nu \tag{16}$$

say ; ν is the ellipticity, here equal to the dynamical ellipticity, produced by the rotation. Then the terms in L arising from the deformation reduce to

$$\tfrac{1}{2}\nu C\left[\dot{l}^2 + \dot{m}^2 + 4\dot{\chi}^2 + \left(2 + \frac{3}{1+\kappa}\right)n(l\dot{m} - \dot{l}m)\right.$$
$$\left. + n^2\left\{\left(1 + \frac{3}{1+\kappa}\right)^2 l^2 + m^2 + \frac{9\chi^2}{(1+\kappa)^2}\right\}\right] \tag{17}$$

The terms in l, m, χ are equivalent to increasing A, B, C by

$$\left.\begin{aligned} \delta A &= \tfrac{1}{3}\nu C + \frac{2\nu C}{1+\kappa} \\ \delta B &= \tfrac{1}{3}\nu C - \frac{\nu C}{1+\kappa} \\ \delta C &= -\tfrac{2}{3}\nu C - \frac{\nu C}{1+\kappa} \end{aligned}\right\} \tag{18}$$

The changes of the differences of the moments of inertia due to the potential Ψ are just those given by these expressions with the sign reversed. If we write $A+\delta A=A_0$ and so on,

$$L=\frac{1}{2}\left\{\left(B_0+\frac{2}{3}\nu C+\frac{\nu C}{1+\kappa}\right)\dot{l}^2+\left(A_0+\frac{2}{3}\nu C-\frac{2\nu C}{1+\kappa}\right)\dot{m}^2+\left(C_0+\frac{8}{3}\nu C+\frac{\nu C}{1+\kappa}\right)\dot{\chi}^2\right\}$$
$$+\frac{1}{2}n\left\{B_0+A_0-C_0+C\nu\left(\frac{8}{3}+\frac{4}{1+\kappa}\right)\right\}(l\dot{m}-\dot{l}m).$$
$$-\frac{1}{2}n^2\left\{l^2(C_0-A_0)+m^2(C_0-B_0)\right\}-\frac{3n^2}{2(1+\kappa)}\left\{l^2(C_0-A_0)+\chi^2(B_0-A_0)\right\} \quad (19)$$

We write

$$C_0-B_0=A_0\alpha\ ;\ C_0-A_0=B_0\beta\ ;\ B_0-A_0=C_0\gamma. \quad (20)$$

The free speeds, where squares of the ellipticities and their products by ν are neglected, are

$$n\left(1+\frac{3\beta}{2(1+\kappa)}+\frac{9\nu}{2(1+\kappa)}\right),\ n\left\{\left(1+\frac{3}{1+\kappa}\right)\alpha\beta\right\}^{1/2},\ n\left(\frac{3\gamma}{1+\kappa}\right)^{1/2}. \quad (21)$$

The two slower speeds are to this accuracy the same as for a rigid body with moments of inertia A_0, B_0, C_0. The speed near n is slightly increased. As an indication of the probable amount, and for further comparisons, we notice that β is about 0·0006, α about 0·0004. If the Moon had its hydrostatic ellipticities β would be about 0·00004, α about 0·000009 ; and ν* is about $1{\cdot}5\times10^{-7}$. The term in ν for the speed near n is therefore of the order of 10^{-3} of the β term. The uncertainty of β as found from observations of the inclination of the axis to the ecliptic is about 1 part in 400. The point may possibly become important if the accuracy should be increased. A speed near n represents a slow motion in space ; and β is determined from a 19-year period, so that a small change in the resonating frequency might matter. The point is analogous to the difference between the effects of elasticity of the Earth on precession and the variation of latitude.

3. *Correction for imperfect elasticity*.—The simplest type of elastic afterworking is the linear form specified by replacing the rigidity μ by the operator (Jeffreys 1952, p. 10)

$$\mu_0\frac{1+1/p\tau'}{1+1/p\tau}, \quad (1)$$

where for a time factor exp$i\sigma t$, p will be replaced by $i\sigma$. For a stress P applied at $t=0$ and kept constant, there is an initial elastic strain P/μ_0, which increases asymptotically to $(P/\mu_0)(\tau'/\tau)$ if the stress is maintained for a time longer than τ'. Since for the Moon gravity has a small effect compared with rigidity we can take ν to vary as $1/\mu$. Since we have the solution for constant ν, say ν_0, we can adapt by putting

$$\nu=\nu_0+\nu_1, \quad (2)$$

$$\nu_1=\nu_0\frac{\tau'-\tau}{\tau(p\tau'+1)}, \quad (3)$$

adding terms in ν_1 to the Lagrangian, and solving the modified period equation by successive approximation. As the data for τ and τ' are very rough in any

* $(\mu/\rho)^{1/2}$ is the velocity of transverse elastic waves; in the greater part of the Earth's shell it ranges from about 4·4 to 7 km/sec. I generally take 4·5 km/sec.

case we may as well drop the small number κ. The time factors for the free modes are respectively

$$\pm in\left(1 + \frac{3}{2}\beta\right) - 2\frac{n^2\nu_0(\tau' - \tau)\tau'}{\tau(n^2\tau'^2 + 1)}, \tag{4}$$

$$\pm in(4\alpha\beta)^{1/2} - 4n^2\alpha\beta\left(\frac{2}{\beta} + \frac{1}{2\alpha}\right)\frac{\nu_0(\tau' - \tau)\tau'}{\tau(4n^2\alpha\beta\tau'^2 + 1)}. \tag{5}$$

$$\pm in(3\gamma)^{1/2} - \frac{9n^2\gamma}{2}\frac{\nu_0(\tau' - \tau)\tau'}{\tau(3\gamma n^2\tau'^2 + 1)} \tag{6}$$

The case $\tau' = \infty$ is that of elasticoviscosity.

The only phenomenon in the Earth's shell that seems to give positive evidence for imperfections of elasticity at small stresses is the damping of the free variation of latitude. There are two types of solution according as $\sigma\tau$, $\sigma\tau'$ are large or small for this speed. If τ, τ' are large (say >60 days) the solution should satisfy (Jeffreys 1952, p. 247, (21). 10^{-8} should be 10^{-9}.)

$$\frac{1}{\theta} = \frac{1}{\tau} - \frac{1}{\tau'} = \frac{1}{2 \times 10^8\,\text{sec.}} = \frac{1}{6\,\text{years}} \tag{7}$$

This may be wrong by a factor of 2. If the term 1 in the denominators of the damping factors is dropped (corresponding to infinite τ'), these factors become simply

$$-\frac{2\nu_0}{\theta}, -\left(\frac{2}{\beta} + \frac{1}{2\alpha}\right)\frac{\nu_0}{\theta}, -\frac{3}{2}\frac{\nu_0}{\theta}. \tag{8}$$

With the numerical values already adopted the times of relaxation are, roughly,

$$2 \times 10^7 \text{ years} \; ; \; 10^4 \text{ years} \; ; \; 3 \times 10^7 \text{ years}. \tag{9}$$

If we take $\tau' = 1{\cdot}1\tau$, we get $\tau' = 0{\cdot}6$ year. The free periods are roughly 1 month, 100 years, and 3 years. Then the first relaxation time is hardly altered ; the others are multiplied by about 600 and by 3.

The alternative solution for $\sigma\tau$, $\sigma\tau'$ small gives

$$\tau' - \tau = 1{\cdot}8 \text{ days} \tag{10}$$

In this case the relaxation times are 10^5, 4×10^7, and 4×10^8 years. The ratio τ'/τ does not affect the results appreciably.

On the face of it the damping in any case is enough to account for the smallness of the free librations if the time since the Moon became solid is comparable with the age of the Earth.

3.1. *Subsidence of a surface inequality.*—There are however two possible difficulties. If the Moon is imperfectly elastic, the ellipticities themselves must be tending to subside and leave the Moon in a hydrostatic state. The differential equation satisfied in a free body for a surface harmonic of degree n is

$$\left(1 + \frac{ng\rho a}{(2n^2 + 4n + 3)\mu}\right)\epsilon = 0,$$

where μ must be replaced by an operator as in (1). For a second harmonic on the Moon this leads to a damping factor

$$\lambda = \left(\frac{1}{\tau'} + \frac{0{\cdot}015}{\tau}\right) \Big/ 1{\cdot}015 \tag{11}$$

With $\tau' = \infty$, $\tau = 6$ years, this would give a time of relaxation of about 400 years ; the inclination of the Moon's axis to the ecliptic would have changed considerably

even during the interval of accurate observation. Elastic afterworking, with the same value of θ, would make the time even shorter. It looks as if the excess ellipticities could not have persisted long enough for the librations to become appreciably damped.

The solution appears to be that the stress-strain relation that leads to (1) assumes that the body is not self-stressed. If there are stress-differences in a state of equilibrium, there is no reason why there should not be a linear relation between stress and strain when both are measured from this equilibrium state. Then the times just found would represent the time-scale of the return to this state. There is even no conclusive reason against using the elasticoviscous relation in this way. On the other hand if we made the usual assumption that change of shape and stress-difference are always measured from the hydrostatic state, and adopted the elasticoviscous law with the viscosity derived from the Fenno-Scandian uplift, the time of relaxation would only be increased to something of the order of 40 000 years, and it would still be impossible to suppose that the ellipticities have existed for anything like the age of the Earth.

3.2. The second difficulty is as follows. The evidence for the Earth points to a strength comparable with that of surface rocks to a depth of order 600 km. The viscosity inferred from the variation of latitude may refer only to the lower part of the mantle. If the difference is due to temperature, we should expect the transition to be at about the same depth in the Moon; but this depth is nearly half the radius. Couples due to tidal friction in a uniform body are proportional to a high power of the radius, and it is possible on this ground that the times should all be multiplied by, say, 2^6. This would bring the longer ones to the neighbourhood of 10^9 years. The initial amplitudes of the librations are of course unknown, but it is reasonable to suppose that those in latitude were once of the order of 5° and that in longitude near 90°; and a factor of $e^{-4\cdot 5}$ would not go far towards reducing these values to their present values, which can hardly be more than $1'$ or so. This is not fatal, since the factor 2^6 is dubious and the viscosity may be wrong by a factor of 2; if the damping has been by e^{-9} it would probably be sufficient.

However, the damping factor contains a factor n^2. If, as I originally supposed, the Moon solidified when much nearer the Earth than it is now, n^2 was much larger. Also the cooling might not have reached so great a depth. On both accounts the damping may have formerly been much more rapid than it is now, and the difficulty disappears.

4. The theory is not difficult to adapt to a freely rotating satellite. Only the effects on l and m arise, since the amplitude and phase of χ are replaced in free rotation by the third Euler angle and its rate of increase; a forced motion is superposed. The results remain of the same order of magnitude so long as the mean motion and the rate of rotation are comparable. The secular effects considered by Darwin are of course of the second order in the tidal potential.

5. *Tidal friction and rotations of satellites.*—In 'The Earth', 8·02 (13), I show that for a homogeneous body the tidal frictional couple is

$$-N = -\frac{8}{5}\pi\frac{fma^4}{c^3}\rho H \sin 2\epsilon, \tag{1}$$

where ρ and a are the density and radius of the disturbed body, m, c the mass and mean distance of the disturbing body, H and 2ϵ the maximum amplitude

and lag of the semidiurnal tide. For a satellite with rotation speed ω and mean motion n

$$H \sin^2\theta \exp\{2i(\chi+\omega t-nt)\} = \frac{5\rho a^3}{19\mu+2g\rho a} \cdot \frac{3fma^2}{4c^3} \sin^2\theta \exp\{2i(\chi+\omega t-nt)\} \tag{2}$$

in which we must write

$$\mu = \mu_0 \frac{1+1/2i(\omega-n)\tau'}{1+1/2i(\omega-n)\tau} \doteqdot \mu_0\left(1-\frac{1}{2i(\omega-n)\theta}\right), \tag{3}$$

provided $(\omega-n)\tau$, $(\omega-n)\tau'$ are large. For the Earth the gravity term is rather small compared with the rigidity term where they are combined, and for smaller bodies it is much smaller. Then effectively H has the value for a perfectly elastic body and for large τ, τ'

$$2\epsilon = \frac{1}{2(\omega-n)\theta}. \tag{4}$$

We write as before

$$fm/c^3 = n^2/(1+\kappa) \doteqdot n^2, \tag{5}$$

and in all the angular acceleration is

$$\dot\omega = -\frac{N}{C} = -\frac{90\pi}{8}\left(\frac{n^2a}{1+\kappa}\right)^2 \frac{\rho}{19\mu_0+2g\rho a}\,\frac{1}{2(\omega-n)\theta}. \tag{6}$$

As trial values for the Moon we take a rotation period of 6 days and a revolution period of 12 days. We then find

$$\dot\omega = 2\times 10^{-19}/\text{sec}^2, \tag{7}$$

and a change of the order of the initial value would need something of the order of 2×10^6 years. At the present distance this time would need to be multiplied by about 20, and if the imperfection of elasticity is confined to the inner half of the radius a further factor of the order of 100 might be needed. It is therefore very unlikely that the Moon's present rotation could have been established unless it was formerly much nearer the Earth than it is now.

We may notice that the differential equation for ω is of the form

$$\dot\omega = -A/(\omega-n), \tag{8}$$

which has the solution

$$(\omega-n)^2 = (\omega_0-n)^2 - 2At. \tag{9}$$

With the above values ω would become equal to n in about 10^6 years. The approximations would however fail when $\omega-n$ is small enough for ϵ to become a large angle, and the approach would become asymptotic ; and at a later stage still the libration case would be reached.

For Mercury, if we take the same initial rotation period, we get a time near 10^9 years. With a larger initial rate of rotation, as is probable, this would be much prolonged. It appears that there is a serious difficulty in accounting for the present rotation of Mercury even if the imperfection of elasticity adopted for the Earth is supposed to hold right up to the surface.

In the case $(\omega-n)\tau$, $(\omega-n)\tau'$ small, the lag

$$2\epsilon = \tan^{-1}\{2(\omega-n)(\tau'-\tau)\} \doteqdot 1 \tag{10}$$

with the values from 3 (10); whereas (4) makes it 1/2000. Thus in this case the value of $\dot\omega$ just given would be multiplied by about 2000. Thus the smaller values of τ, τ' afford adequate tidal friction to account for the rotations of the Moon and Mercury without further adjustment.

6. *Other satellites.*—Far the most important factor in $\dot{\omega}$ for satellites is $(n^2a)^2$, apart from hitherto unknown variations in θ. It is therefore of interest to tabulate rough values; I give (diameter/period^2)÷(km/day^2). For most of the diameters of Saturn's satellites I use results communicated to me in a letter from Professor Kuiper; for the rest I take those given by Russell, Dugan and Stewart.

Moon	5	Mimas	560	Ariel	80
Phobos	160	Enceladus	270	Umbriel	22
Deimos	6	Tethys	230	Titania	13
JV	600	Dione	110	Oberon	5
JI	1200	Rhea	80		
JII	300	Titan	18	Triton	130
JIII	100	Hyperion	0·9		
JIV	18	Iapetus	0·2		
JVI	0·002	Phoebe	0·001		

The great majority of satellites' rotations should be more sensitive to tidal friction than that of the Moon. It is therefore not surprising that all whose rotations are known keep a constant face to their primaries except those of Uranus (with a possibility of a very small difference for JIII to which Dr W. H. Steavenson has called my attention). The best-authenticated case of all, however, is Iapetus, for which the effect should be much less than for the Moon.

Dr Steavenson informs me that Titania and Oberon have axes not normal to the orbital planes and possibly in them. There are cases where this state is theoretically stable, as Stratton (1906) has shown, but the physical properties required are very different from those that appear to belong to the Earth. There would be nothing surprising in this, since the small densities of some of the satellites of the outer planets would already suggest very different constitutions, possibly largely ice at very low temperatures. But it seems worth while to state the result that the rotations of Iapetus and the satellites of Uranus provide additional evidence.

7. The law of elasticoviscosity, proposed by Maxwell and occasionally used by Darwin, was originally based on a model in which, under constant stress, the molecules break down at a rate proportional to the stress and permit flow at a steady rate superposed on the initial elastic yield. In elastic afterworking a steady stress beginning at a certain time produces an initial elastic yield, but the strain increases asymptotically to a new constant value. This behaviour is represented in the simplest possible way by the law 3(1). Qualitatively it has experimental support for some rocks, but in laboratory conditions the stresses have to be rather large and the response is not linear. In the present problems all the stress-differences that concern us are so small that we are entitled to assume a linear law. It has been shown that if a perfectly elastic material encloses pockets of an elasticoviscous one, the response as a whole can follow the law suggested.

A perfectly elastic body can be in equilibrium under internal stress without external force; a rubber ring gripping a rod is a simple example. It would be quite possible to have a perfectly elastic solid with such internal stress initially, and containing elasticoviscous pockets. In such a body the law for elastic afterworking would hold, the relation being now between departures of stress and strain from their initial values. The elasticoviscous connection between them could then be regarded as the formal limit when τ' is made infinite. I know

of no suitable model for this case, but there seems to be no harm in considering the possibility.

8. The results for the Moon and Mercury with the larger values of τ, τ' are none too satisfactory. Application of the imperfection of elasticity derived from the variation of latitude to the Moon in the most straightforward way indicates that the rotation could easily have been brought into coincidence with the revolution and the free librations damped out. But, if, as is probable, it is effective for only the inner half of the Moon's radius, the damping would be much slower and it is not obvious whether the time available would be enough. If the Moon was formerly very much nearer the Earth the difficulty disappears.

For Mercury similar analysis suggests that if the original rotation was comparable with that of the Earth and Mars, and τ, τ' are long, the time needed to reduce its rotation to the present value would be much too long. We cannot avoid the difficulty by supposing Mercury to have formerly been much nearer the Sun. If it was once a satellite of Venus there is no difficulty in accounting for the slow rotation, but the time needed for it to have receded so far that the satellite orbit became unstable would probably be over 10^{11} years. It might be necessary to suppose that imperfection of elasticity has at some time been greater in Mercury than in the Earth. This would indicate internal temperatures higher than at present. On the other hand, if the smaller values of τ, τ' are correct, all these difficulties disappear.

The experiments of D. W. Phillips suggested $\tau'/\tau = 1{\cdot}6$ at rather large stresses, the new steady state being reached in some weeks. For small stresses $1{\cdot}1$ is likely to be an overestimate, but as it leads to τ about 18 days for the smaller solution it is possibly not wildly wrong. Further, it is likely to be applicable to cold materials, and there may be no reason to suppose that the dissipation is confined to the central regions. The indications, especially for Mercury, are that the solution 5(10) leads to fewer difficulties than 5(4) does.

The satellites of the outer planets are probably largely ice and the rigidity is probably much lower than for rocks ; with equal times of relaxation (τ, τ') this alone would greatly increase the effects of tidal friction, and the imperfection of elasticity may also be greater. If so, the apparent smallness of the theoretical effect on Iapetus may well be compensated.

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FINITE ATMOSPHERES WITH ISOTROPIC SCATTERING

III. Corrigendum and Addendum

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Summary

The conclusions of the Appendix of Paper I are wrong. A rigorous treatment has established the correctness of all the main formulae and it has led to some new and important relations. These are summarized, and they are used to re-establish any results depending on the Appendix.

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1. *Corrigendum.*—Hopf has proved in (3), Section 21, that there is no non-null, bounded solution of the homogeneous equation

$$J(\tau)=\tfrac{1}{2}\varpi\int_0^{\tau_1} J(t)E_1(|t-\tau|)dt \quad (0<\varpi\leqslant 1), \tag{1.1}$$

i.e., in the notation of Paper I (see (1)), of the equation

$$J(\tau)=\varpi\bar{\Lambda}_\tau\{J(t)\}.$$

It is not difficult to go further and to prove that, if $J(\tau)$ is $O(ln\tau^{-1})$ as $\tau\to+0$ and $O(ln[\tau_1-\tau]^{-1})$ as $\tau\to\tau_1-0$, then $J(\tau)\equiv 0$. These results contradict the conclusions of the Appendix to Paper I.

In the conservative case the error is due to the non-uniqueness of the solutions of the X- and Y-equations, so that the argument cannot be reversed. In the non-conservative case it is due to the fact that $X(\mu)$ and $Y(\mu)$ have no poles. (See Section 2, below).

In the main part of Paper I, *all the final results are correct.* This is even true when, as in Sections 8 and 10 (conservative case), they are apparently based on the Appendix. In the non-conservative case the solution of (1.1) has been taken to be zero and, provided that $X(\mu)$ and $Y(\mu)$ are defined by I(4.21), the analysis of these sections can be justified rigorously. In the conservative case the analysis has to be altered. After summarizing some new and important formulae (Section 2), the revised analysis is given in Sections 3 and 4.

2. *Addendum.*—If there is no non-null solution of the equation (1.1), the only solution of the auxiliary equation

$$J(\tau,\sigma)=\varpi\bar{\Lambda}_\tau\{J(t,\sigma)\}+\exp(-\sigma\tau) \tag{2.1}$$

is the N-solution (obtained by iteration), viz.

$$J(\tau,\sigma)=\sum_{n=0}^{\infty}\varpi^n\bar{\Lambda}_\tau{}^n\{\exp(-\sigma t)\}. \tag{2.2}$$

Since

$$\bar{\Lambda}_\tau\{1\}=1-\tfrac{1}{2}E_2(\tau)-\tfrac{1}{2}E_2(\tau_1-\tau),$$

therefore

$$\bar{\Lambda}_\tau\{1\}\leqslant 1-E_2(\tfrac{1}{2}\tau_1)=\rho<1. \tag{2.3}$$

It follows that, if $f(\tau)$ is continuous for $0 \leqslant \tau \leqslant \tau_1$, so that $|f(\tau)| \leqslant K$ (say) in $(0, \tau_1)$, and if $0 < \varpi \leqslant 1$, then†

$$|\varpi^n \bar{\Lambda}_\tau{}^n\{f(t)\}| \leqslant K\bar{\Lambda}_\tau{}^n\{1\} \leqslant K\rho^n. \tag{2.4}$$

Hence $\sum_{n=0}^{\infty} \varpi^n \bar{\Lambda}_\tau{}^n\{f(t)\}$ is (absolutely) convergent for $0 < \varpi \leqslant 1, 0 \leqslant \tau \leqslant \tau_1$. It follows that the series (2.2) converges for $0 < \varpi \leqslant 1$, $0 \leqslant \tau \leqslant \tau_1$ and any finite σ.

Let $X(\mu)$ and $Y(\mu)$ be defined by I(4.21), viz.

$$X(\mu) = J(0, \mu^{-1}), \quad Y(\mu) = J(\tau_1, \mu^{-1}). \tag{2.5}$$

Then $X(\mu)$ and $Y(\mu)$ are positive and continuous for $\mu \neq 0$ and it is easy to show that

$$X(\mu) \to 1, \quad Y(\mu) \to 0 \qquad \text{as } \mu \to +0 \tag{2.6}$$

$$X(\mu) \to J(0,0), \quad Y(\mu) \to J(0,0) \qquad \text{as } |\mu| \to \infty. \tag{2.7}$$

The equality of these limits as $|\mu| \to \infty$ is most easily seen from the relation I(4.26), viz.

$$Y(\mu) = \exp(-\tau_1/\mu) X(-\mu), \tag{2.8}$$

together with the fact that $J(\tau, \sigma)$ is a continuous function of σ at $\sigma = 0$.

From (2.2), we have

$$J(\tau, \sigma) = \sum_{n=0}^{\infty} \varpi^n \bar{\Lambda}_\tau{}^n \left\{ \sum_{m=0}^{\infty} \frac{(-\sigma t)^m}{m!} \right\} = \sum_{m=0}^{\infty} \frac{(-\sigma)^m}{m!} \sum_{n=0}^{\infty} \varpi^n \bar{\Lambda}_\tau{}^n \{t^m\}.$$

The inversions of the orders of summation and integration can all be justified by absolute convergence on using (2.4). Putting $\tau = 0$ and τ_1, and $\sigma = 1/\mu$, we have the expansions (valid for $|\mu| > 0$)

$$X(\mu) = \sum_{m=0}^{\infty} \frac{(-1)^m C_m}{m!} \mu^{-m}, \quad Y(\mu) = \sum_{m=0}^{\infty} \frac{(-1)^m D_m}{m!} \mu^{-m}, \tag{2.9}$$

where

$$C_m = \left[\sum_{n=0}^{\infty} \varpi^n \bar{\Lambda}_\tau{}^n \{t^m\} \right]_{\tau=0}, \tag{2.10}$$

$$D_m = \left[\sum_{n=0}^{\infty} \varpi^n \bar{\Lambda}_\tau{}^n \{t^m\} \right]_{\tau=\tau_1}. \tag{2.11}$$

Relations between the coefficients C_m, D_m, the moments of the X- and Y-functions (see I(7.14)) and τ_1 are found by substituting into I(4.22) and I(4.26), expanding in powers of μ^{-1} and equating coefficients. It is found, in particular, that

$$C_0 = D_0 = J(0,0) = 1/(1 - x_0 + y_0). \tag{2.12}$$

The expansions (2.9), which converge for $|\mu| > 0$, show that $X(\mu)$ *and* $Y(\mu)$ *are analytic functions of the complex variable* μ *which are regular for* $|\mu| > 0$ (*infinity included*).

Conservative case ($\varpi = 1$). In this case it can be shown that the moments of $X(\mu)$ and $Y(\mu)$ satisfy the relations

$$x_0 + y_0 = 1, \tag{2.13}$$

$$x_1 - y_1 = \tau_1 y_0, \tag{2.14}$$

$$y_0[x_2 + y_2 + \tfrac{1}{2}\tau_1(x_1 + y_1)] = \tfrac{1}{6}. \tag{2.15}$$

† Since the kernel of $\bar{\Lambda}$ is positive, if $g(\tau) \leqslant h(\tau)$ for $0 \leqslant \tau \leqslant \tau_1$, then $\bar{\Lambda}_\tau\{g(t)\} \leqslant \bar{\Lambda}_\tau\{h(t)\}$. The inequalities in (2.4) follow by repeated application of this result.

Since $J(0,0)$ is finite and not zero, it follows from (2.12) and (2.13) that $x_0 \neq 1$, $y_0 \neq 0$. Thus the functions defined by (2.5) are not the standard solutions.

By the method indicated above, it can be shown, after some lengthy calculations, that

$$C_0 = D_0 = 1/2y_0, \tag{2.16}$$

$$C_1 = \tfrac{1}{2}(\tau_1 C_0 - \gamma), \quad D_1 = \tfrac{1}{2}(\tau_1 C_0 + \gamma), \tag{2.17}$$

where

$$\gamma = \tfrac{1}{2}\tau_1{}^3 y_0 + 6(x_3 - y_3) + 3\tau_1(x_2 - y_2). \tag{2.18}$$

3. *Section* 8 *of Paper* I.—With the X- and Y-functions defined by (2.5), the analysis of Section 7 ($0 < \varpi < 1$) of Paper I holds also when $\varpi = 1$ up as far as I(7.18), viz. as far as the equations

$$J^0(0)(1 - x_0) + J^0(\tau_1)y_0 = a_0,$$

$$J^0(0)y_0 + J^0(\tau_1)(1 - x_0) = a_0.$$

These are identical, since $x_0 + y_0 = 1$, and we have only one equation

$$J^0(0) + J^0(\tau_1) = a_0/y_0. \tag{3.1}$$

To get a second equation, we can use I(8.18) with

$$I(0, +\mu) = j_0{}^0(\mu^{-1}) = J^0(0)X(\mu) - J^0(\tau_1)Y(\mu), \tag{3.2}$$

$$I(\tau_1, -\mu) = j_1{}^0(\mu^{-1}) = J^0(\tau_1)X(\mu) - J^0(0)Y(\mu), \tag{3.3}$$

and $f_0(\mu) = f_1(\mu) = 0$, $B_1(\tau) = a_0$. Then we get

$$[J^0(\tau_1) - J^0(0)][x_2 + y_2 + \tfrac{1}{2}\tau_1(x_1 + y_1)] = 0.$$

By (2.15) the second bracket is not zero, and hence

$$J^0(0) = J^0(\tau_1) = a_0/2y_0. \tag{3.4}$$

Alternatively, we can use the fact that

$$s^{-1}j_0{}^n(s) \equiv \int_0^{\tau_1} J^n(t)\exp(-st)\,dt \qquad (n = 0, 1, \ldots) \tag{3.5}$$

is an integral function and therefore regular at $s = 0$. [For $J^n(\tau)$ is the N-solution of I(7.2) with $\varpi = 1$ and therefore

$$J^n(\tau) = \sum_{n=0}^{\infty} \bar{\Lambda}_\tau{}^n \left\{ \frac{a_0 t^n}{n!} + \frac{a_1 t^{n-1}}{(n-1)!} + \ldots + a_n \right\}.$$

By (2.4) this converges for $0 \leqslant \tau \leqslant \tau_1$, and $|J^n(\tau)|$ is bounded in $(0, \tau_1)$ by

$$(|a_0|\tau_1{}^n/n! + \ldots + |a_n|)/(1 - \rho).$$

Since, by the smoothing property of $\bar{\Lambda}$, $J^n(\tau)$ is continuous for $0 < \tau < \tau_1$, it follows from a well-known theorem in analysis that $s^{-1}j_0{}^n(s)$ is an integral function.]

By (2.9), (2.12) and (3.2),

$$s^{-1}j_0{}^0(s) = C_0[J^0(0) - J^0(\tau_1)]s^{-1} + \sum_{m=1}^{\infty}(-1)^m[C_m J^0(0) - D_m J^0(\tau_1)]\frac{s^{m-1}}{m!},$$

and this can contain no negative power of s. Hence $J^0(0) = J^0(\tau_1)$ and thus we get (3.4).

When $n = 1$, as in section 7 of Paper I, we get one independent equation, viz.

$$J^1(0)(1 - x_0) + J^1(\tau_1)y_0 - J^0(0)x_1 + J^0(\tau_1)y_1 = a_1.$$

By (2.13), (3.4) and (2.14), this can be written

$$y_0[J^1(0)+J^1(\tau_1)] = \tfrac{1}{2}a_0\tau_1 + a_1. \tag{3.6}$$

To obtain a second equation we can either use I(8.18) as above, or we can expand

$$\begin{aligned} s^{-1}j_0{}^1(s) = s^{-1}X(s^{-1})[J^0(0)s^{-1}+J^1(0)] \\ -s^{-1}Y(s^{-1})[J^0(\tau_1)s^{-1}+J^1(\tau_1)], \end{aligned} \tag{3.7}$$

which comes from I(7.10) with $n=1$, in ascending powers of s and equate to zero the coefficients of the negative powers. On using (3.4) we get

$$J^1(0)-J^1(\tau_1) = a_0(C_1-D_1) = -a_0\gamma, \tag{3.8}$$

where γ is given by (2.18). From (3.6) and (3.8), we have

$$\left.\begin{aligned} J^1(0) &= \{a_0(\tfrac{1}{2}\tau_1 - y_0\gamma)+a_1\}/2y_0, \\ J^1(\tau_1) &= \{a_0(\tfrac{1}{2}\tau_1 + y_0\gamma)+a_1\}/2y_0. \end{aligned}\right\} \tag{3.9}$$

Thus, when $\varpi=1$, the formulae I(7.21) and I(7.22) become

$$\begin{aligned} I(0,+\mu) = {} & \frac{X(\mu)}{2y_0}\{a_0\mu+a_1+a_0(\tfrac{1}{2}\tau_1-y_0\gamma)\} \\ & -\frac{Y(\mu)}{2y_0}\{a_0\mu+a_1+a_0(\tfrac{1}{2}\tau_1+y_0\gamma)\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} I(\tau_1,-\mu) = {} & \frac{X(\mu)}{2y_0}\{-a_0\mu+a_1+a_0(\tfrac{1}{2}\tau_1+y_0\gamma)\} \\ & -\frac{Y(\mu)}{2y_0}\{-a_0\mu+a_1+a_0(\tfrac{1}{2}\tau_1-y_0\gamma)\}. \end{aligned} \tag{3.11}$$

The number γ, can be calculated from (2.18) or from the relation

$$\gamma = D_1 - C_1 = \left[\sum_{n=0}^{\infty}\bar{\Lambda}_\tau{}^n\{t\}\right]_{\tau=\tau_1} - \left[\sum_{n=0}^{\infty}\bar{\Lambda}_\tau{}^n\{t\}\right]_{\tau=0}. \tag{3.12}$$

To change from the X- and Y-functions used here to the standard functions $X^s(\mu)$, $Y^s(\mu)$ with moments $x_n{}^s$, $y_n{}^s$, such that

$$x_0{}^s = 1, \quad y_0{}^s = 0, \tag{3.13}$$

we have to write

$$X(\mu) = X^s(\mu) - y_0\mu[X^s(\mu)+Y^s(\mu)]/(x_1{}^s+y_1{}^s), \tag{3.14}$$

$$Y(\mu) = Y^s(\mu) + y_0\mu[X^s(\mu)+Y^s(\mu)]/(x_1{}^s+y_1{}^s) \tag{3.15}$$

in (3.10) and (3.11). After a lengthy reduction the equations I(8.11) and I(8.12) are obtained, the constants $J^1(0)$, $J^1(\tau_1)$ being given in terms of the moments $x_n{}^s$, $y_n{}^s$ by I(8.19)–(8.21). On comparing the results of Paper I, Section 8, with (3.10) and (3.11) it will be seen that it is much simpler to work with the X- and Y-functions defined by (2.5) than with the standard functions. Unfortunately it is the latter which have been tabulated.

4. *Section* 10 *of Paper I.*—When $\varpi=1$, the analysis of Section 9 holds as far as I(9.12), which becomes

$$\Phi^2(0)+\Phi^2(\tau_1) = 1. \tag{4.1}$$

By expanding $s^{-1}\Phi_0{}^2(s)$ in ascending powers of s and equating the coefficient of s^{-1} to zero, we get

$$\Phi^2(0)-\Phi^2(\tau_1) = 1-2y_0. \tag{4.2}$$

Hence

$$\Phi^2(0) = 1-y_0 = x_0, \quad \Phi^2(\tau_1) = y_0, \tag{4.3}$$

and these are I(9.14). Thus the equations I(9.15)-(9.18) hold also when $\varpi = 1$, if $X(\mu)$ and $Y(\mu)$ are defined by (2.5). On changing to the standard functions, we get the results of Section 10.

Concluding note.—It has been possible to give only a brief indication of the methods by means of which results can be proved rigorously. The treatment is based upon N-solutions ; there are few difficulties and the analysis is surprisingly simple and elegant. The need for complete rigour is shown by my own mistake.

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1957 *July* 9.

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THE EFFECT OF REFLECTION ON THE DETERMINATION OF MASSES OF CLOSE BINARY SYSTEMS

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Summary

The " reflection effect " in close binary systems is briefly described, with particular reference to the errors it may introduce into the determination of masses and mass-ratios of spectroscopic binary systems.

Recent work on the photometric effects of reflection in systems in which both stars are sensibly spherical and revolve in circular orbits is surveyed, and its results extended and adapted to the purpose of estimating the correction which should be applied to the velocity curves of such systems at any phase. To this end, it is necessary to consider the contribution to the observed radial velocity made by the rotation of the reflecting star. This is considered for a star which rotates about an arbitrarily inclined axis and with an arbitrary angular velocity, the only restriction being that the star must rotate as a rigid body.

From these results a phase law for the effect of reflection on the radial velocity of a binary component is derived. From this law, the correction to be made to the observed masses is derived in terms of the luminosities, radii and separation of the two components. In deriving this correction the effect of any " spurious eccentricity " introduced by distortion of the velocity curve by the phase law is ignored. Finally the magnitude of the correction is roughly estimated for the case in which each star rotates with the Keplerian angular velocity about an axis perpendicular to the orbital plane. For most of the systems for which the theory of this investigation is valid the total correction to the *total* mass of the system would seem to be ~ 10 per cent of the mass of the heavier component. By extrapolation to closer systems, however, it would seem that much larger corrections may sometimes be expected.

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1. *Introduction.*—The so-called "reflection effect" in close binary systems is by now well known, one of the earliest comprehensive treatments of it being that of Eddington (1) in 1926. The term " reflection " is to some extent a misnomer, the processes involved being absorption and re-emission in an unknown combination with scattering. Nonetheless, "reflection" is a descriptive term, indicating that some of the light of each component is intercepted by the other, and re-emitted in the direction of the observer. Thus, the hemispheres of the two stars which face each other are brighter than the rest of their respective surfaces, and differing amounts of the bright hemispheres are revealed with changing phase producing photometric effects with which all observers of these systems are familiar. The law controlling the " reflection " is not accurately known, but Kopal has shown in (2) that Lambert's Law is at least a reasonable approximation, and on the basis of this law he has recently made a thorough investigation (3) of the photometric effect of reflection which takes into account the finite radii of both stars and is accurate to the extent that both stars can be regarded as spheres; that is to say, to terms of the fourth power of their ratio of their radii to their separation.

Besides the effect of reflection on light curves, however, there is also an effect on velocity curves. This arises because the distribution of light over the apparent

disk of the star no longer possesses radial symmetry and so the velocity measured by the spectrograph is not that of the centre of gravity of the star. The light centre is displaced from the geometrical centre of the disk into the bright hemisphere, and once the radial symmetry of the light distribution is destroyed, the axial rotation of the star contributes to the observed radial velocity. The axial rotation of the star may increase or diminish the observed velocity according as the star rotates in a direct or retrograde fashion (though the latter has never been observed), and this correction will depend on the angular velocity and inclination of the axis of rotation. It is the purpose of this investigation to discover the magnitude of the correction for this effect and so to obtain some idea of the error in the estimation of the mass of the system.

This object has already attracted a number of other investigators. Some time ago Kopal (**4**) examined this effect, considering the illuminating star as a point source. In 1954–5, Kitamura (**5, 6**) published two papers in which the finite size of the illuminating star was taken into account. However, he derived a correction of only one or two per cent to the masses of *u* Her and μ^1 Sco, where Kopal derived corrections of up to six per cent. Recently, Ovenden (**7**) has suggested that, in some systems, corrections of the same order as the observed mass itself should be applied. It seems, therefore, highly desirable that a thorough investigation of the range of magnitude of this effect should be made, and, as a beginning to such an investigation, we have considered the effect under the following restrictions: both stars are to be regarded as spherical, and to be revolving in circular orbits; no restriction will be placed on the inclination of the axis of rotation of either star, or on their speeds of rotation, except that we are precluded by the first condition from considering very fast-rotating stars, and it will be assumed that the stars rotate as rigid bodies. The law of "reflection" will be assumed to be Lambert's law. With these assumptions we shall be able to use many of the results obtained by Kopal in (**3**)—which will be quoted in the next section—to obtain a phase law which could, in principle, be used to rectify velocity curves before orbital solutions are made, just as light curves are rectified. If the spectroscopic binary is also an eclipsing binary, the phase law will not apply during eclipses—this situation presents a somewhat different problem, which has recently received an elegant treatment at the hands of Hosokawa (**8**). We shall not be able to apply our results to such close systems as Ovenden considered, as the assumption that the stars remain spherical would not be valid. We may, however, obtain some indication of the order of magnitude of the correction to be applied, even for these systems.

2. *Summary of previous results needed in this investigation.*—In all that follows, unless stated otherwise, the suffix 1 refers to the illuminating star, and 2 to the reflecting star, L, m, r stand for the luminosities, masses and radii, respectively, of the two stars, R is their separation, i denotes the inclination of the orbital plane to the celestial sphere, and ψ is the phase angle measured from superior conjunction of the reflecting star. The quantity ϵ will also be useful, where

$$\cos\epsilon = \cos\psi \sin i.$$

The basic equation giving the correction $\Delta v(\epsilon)$ to be added to the observed radial velocity has already been given by Kopal in (**4**). It is

$$\Delta v(\epsilon) = -\frac{\int_{\Sigma(\epsilon)} V_R(\phi, \eta)\mathscr{I}(\phi, \eta)d\sigma}{L_2 + L_R(\epsilon)} \qquad (1)$$

where $\Sigma(\epsilon)$ is the total visible reflecting surface, and $d\sigma$ is the element of this surface. The difference in radial velocity between the point (ϕ, η) and the geometrical centre of the disk is given by $V_R(\phi, \eta)$, and $\mathscr{I}(\phi, \eta)$ is the intensity of the reflected light at the point (ϕ, η). The coordinate system (ϕ, η) will be more clearly defined presently. The total amount of light reflected by the reflecting star at the modified phase angle ϵ is $L_R(\epsilon)$. The minus sign adjusts our definition of ϵ to the convention that velocities of approach are considered negative. Kopal also showed in (4) that $\Delta v(\epsilon)$ is independent of the effective wave-length of observation, if L refers to the *bolometric* luminosities of the two stars.

It is clear that a knowledge of the quantities $V_R(\phi, \eta)$, $\mathscr{I}(\phi, \eta)$ and $L_R(\epsilon)$ and of the nature of the integral operator $\int_{\Sigma(\epsilon)} ()d\sigma$ will provide us with the solution of the problem, and all these, except the first, may be found in Kopal's investigation (3). His results will be quoted without proof here.

First we must define the coordinate system to which the coordinates (ϕ, η) refer. They are spherical polar coordinates defined in the usual way by

$$x = r\mu_1, \quad y = r\mu_2, \quad z = r\mu_3$$

where

$$\mu_1 = \cos\phi \sin\eta, \quad \mu_2 = \sin\phi \sin\eta, \quad \mu_3 = \cos\eta \tag{2}$$

and where the origin is the centre of the reflecting star, while the x-axis always points to the centre of the illuminating star. The z-axis is perpendicular to both the x-axis and the line of sight, and the y-axis is defined so that its positive direction points directly away from the observer for $\epsilon = 90°$.

In this coordinate system, Kopal has shown that for any point (ϕ, η) from which the whole of the illuminating star is visible (the " full-light " zone) $\mathscr{I}(\phi, \eta)$ is given by

$$\mathscr{I}(\phi, \eta) = \frac{L_1}{\pi R^2}\left\{P_1(\mu_1) + \frac{2r_2}{R} P_2(\mu_1) + \frac{3r_2^2}{R^2} P_3(\mu_1) \;.\;.\;.\right\} \tag{3}$$

correctly to terms of the order $(r_2/R)^4$, where the P_n's are Legendre polynomials. For any point (ϕ, η) at which only a part of the illuminating star is visible (the " penumbral " zone) we have

$$\mathscr{I}(\phi, \eta) = \frac{3(1-u_1)}{3-u_1} \mathscr{I}^U(\phi, \eta) + \frac{2u_1}{3-u_1} \mathscr{I}^D(\phi, \eta) \tag{4}$$

where u is the coefficient of limb darkening, and

$$\mathscr{I}^U(\phi, \eta) = \frac{r_1 L_1}{\pi R^3} \sum_{n=0}^{4} C_n^U \left(\frac{R\mu_1}{r_1}\right)^n \tag{5}$$

and

$$\mathscr{I}^D(\phi, \eta) = \frac{r_1 L_1}{\pi R^3} \sum_{n=0}^{4} C_n^D \left(\frac{R\mu_1}{r_1}\right)^n \tag{6}$$

where, again

$$\left.\begin{aligned}
C_0^U &= \frac{2}{3\pi} - \frac{1}{2}\left(\frac{r_2}{r_1}\right) + \frac{1}{\pi}\left(\frac{r_2}{r_1}\right)^2 - \frac{1}{12\pi}\left(\frac{r_2}{r_1}\right)^4 \\
C_1^U &= \frac{1}{2} - \frac{2}{\pi}\left(\frac{r_2}{r_1}\right) + \frac{1}{3\pi}\left(\frac{r_2}{r_1}\right)^3 \\
C_2^U &= \frac{1}{\pi} - \frac{1}{2\pi}\left(\frac{r_2}{r_1}\right)^2 \\
C_3^U &= + \frac{1}{3\pi}\left(\frac{r_2}{r_1}\right) \\
C_4^U &= - \frac{1}{12\pi}
\end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} C_0^D &= \frac{3}{16} - \frac{1}{2}\left(\frac{r_2}{r_1}\right) + \frac{3}{8}\left(\frac{r_2}{r_1}\right)^2 - \frac{1}{16}\left(\frac{r_2}{r_1}\right)^4 \\ C_1^D &= \frac{1}{2} - \frac{3}{4}\left(\frac{r_2}{r_1}\right) + \frac{1}{4}\left(\frac{r_2}{r_1}\right)^3 \\ C_2^D &= \frac{3}{8} - \frac{3}{8}\left(\frac{r_2}{r_1}\right)^2 \\ C_3^D &= +\frac{1}{4}\left(\frac{r_2}{r_1}\right) \\ C_4^D &= -\frac{1}{16} \end{aligned}\right\} \tag{8}$$

all correct to our order of accuracy.

The problem of determining the total amount of reflected light thus reduces to that of integrating powers of μ_1 over the total amount of reflecting surface visible at any time. Again, Kopal has shown in (3) that, except during eclipses, this can be done, in the full light zone, by applying the integral operator $K(\sin\eta_1)$ to the powers of μ_1 and in the penumbral zone by applying the operator $K(\sin\eta_2) - K(\sin\eta_1)$ where

$$K(\sin\eta_1)\{\mu_1^p\} = r_2^2 \int_{\epsilon-\pi/2}^{\cos^{-1}\sin\mu_1} \cos(\epsilon-\phi)\cos^p\phi\left(\int_{\sin^{-1}(\sin\eta\ \sec\phi)}^{\pi-\sin^{-1}(\sin\eta_1\sec\phi)} \sin^{p+2}\eta\, d\eta\right)d\phi \tag{9}$$

and

$$\sin\eta_1 = \frac{r_2+r_1}{R}, \quad \sin\eta_2 = \frac{r_2-r_1}{R},$$

η_1 and η_2 being, respectively, the angle made by the inner and outer sets of common tangents with the line joining the centres of the two stars.

Finally, by applying this result to the expressions in equations (3)–(6), and duly combining the completed integrals we obtain

$$\begin{aligned} L_R(\epsilon) = L_1 \Bigg\{ & \frac{2}{3}\left(\frac{r_2}{R}\right)^2 \frac{(\pi-\epsilon)\cos\epsilon + \sin\epsilon}{\pi} \\ & + \frac{1}{8}\left(\frac{r_2}{R}\right)^3 (3\cos^2\epsilon + 2\cos\epsilon - 1) \\ & + \left(\frac{r_2}{R}\right)^4 \frac{\sin\epsilon\cos^2\epsilon}{\pi} + \left(\frac{r_1 r_2}{R^2}\right)^2 \left[\frac{12}{5\pi^2}\cdot\frac{5+(\pi-5)u_1}{3-u_1} - \frac{1}{\pi}\right]\sin\epsilon \Bigg\}. \end{aligned} \tag{10}$$

This completes the survey of previous results necessary for the present investigation.

3. *The distribution of radial velocity over the surface of the reflecting star.*—We wish now to investigate the nature of the function $V_R(\phi, \eta)$ which is the difference at any point (ϕ, η) between the observed radial velocity at that point and that of the orbital motion of the centre of gravity of the star. As has already been made clear, this difference arises from the axial rotation of the star. We shall suppose the reflecting star to have an angular velocity of rotation $n_2\omega$, where ω is the orbital angular velocity, and n_2 is any positive number. Since we permit the axis of rotation to be arbitrarily inclined to the orbital plane, there is no need to consider the possibility of retrograde rotation ($n_2 < 0$) as retrograde rotation about one pole can always be represented as direct about the other.

It is impossible to specify directly the inclination of the axis of rotation with respect to the coordinate system of the previous section since all three axes change their direction with changing ϵ. Consequently, we introduce a new set of axes $x'y'z'$ having the same origin as the previous set, and whose x'-axis coincides always with the x-axis, but whose z'-axis is the normal to the orbital plane through the centre of the reflecting star; the y'-axis is now defined uniquely. The angle ζ between the z- and z'-axes (or between the y- and y'-axes) changes uniformly with phase, as the z-axis traces a great circle on the surface of the reflecting star, which is inclined at the angle i to the intersection of the orbital plane with the star's surface. We define the positive directions of both the z- and z'-axes so that $\zeta = 90^\circ - i$ when $\psi = 90^\circ$; we shall then obtain, by the standard formulae of spherical trigonometry

$$\sin\zeta = \frac{\sec\psi}{\sqrt{(1+\tan^2\psi\sec^2 i)}}, \quad \cos\zeta = \frac{\tan\psi\tan i}{\sqrt{(1+\tan^2\psi\sec^2 i)}}. \tag{11}$$

The transformation between any two vectors $\mathbf{x}$ and $\mathbf{x}'$ is now given by

$$\mathbf{x} = A\mathbf{x}' \tag{12}$$

where

$$A \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & -\sin\zeta \\ 0 & \sin\zeta & \cos\zeta \end{bmatrix}.$$

In the $x'y'z'$ coordinate system we shall construct a system of spherical polar coordinates (ϕ', η') precisely analogous to the (ϕ, η) coordinates of section 2 and we will now define the position of the pole of rotation by the equations

$$\phi' = \Phi, \quad \eta' = \mathrm{H}, \quad \psi = 0$$

so that at any phase ψ the coordinates of the pole in the (ϕ', η') system are $[(\Phi - \psi), \mathrm{H})$ and the components of angular velocity of rotation about the x', y' and z' axes are, respectively,

$$\begin{aligned} \omega_x' &= n_2\omega\cos(\Phi-\psi)\sin\mathrm{H} \\ \omega_y' &= n_2\omega\sin(\Phi-\psi)\sin\mathrm{H} \\ \omega_z' &= n_2\omega\cos\mathrm{H} \end{aligned} \tag{13}$$

from which, by means of equation (13) it follows that the components about the x, y and z axes are, similarly

$$\begin{aligned} \omega_x &= n_2\omega\cos(\Phi-\psi)\sin\mathrm{H} \\ \omega_y &= n_2\omega\sin(\Phi-\psi)\sin\mathrm{H}\cos\zeta - n_2\omega\cos\mathrm{H}\sin\zeta \\ \omega_z &= n_2\omega\sin(\Phi-\psi)\sin\mathrm{H}\sin\zeta + n_2\omega\cos\mathrm{H}\cos\zeta. \end{aligned} \tag{14}$$

If we denote the linear velocities arising from these three components as V_x, V_y, V_z the radial velocity V_R of each point on the surface is given by

$$V_R = -V_x\cos\epsilon + V_y\sin\epsilon \tag{15}$$

where V_x, V_y are given by the well known equations

$$\begin{aligned} V_x &= z\omega_y - y\omega_z \\ V_y &= x\omega_z - z\omega_x. \end{aligned}$$

Equations (14) and (15) can then be combined to give us

$$\begin{aligned} V_R(\phi,\eta) = n_2 r_2\omega\,[&\cos\phi\sin\eta\{\sin\epsilon\sin(\Phi-\psi)\sin\mathrm{H}\sin\zeta + \cos\mathrm{H}\cos\zeta\sin\epsilon\} \\ &+ \sin\phi\sin\eta\{\cos\epsilon\sin(\Phi-\psi)\sin\mathrm{H}\sin\zeta + \cos\epsilon\cos\mathrm{H}\cos\zeta\} \\ &+ \cos\eta\{-\cos\epsilon\sin(\Phi-\psi)\sin\mathrm{H}\cos\zeta + \cos\epsilon\cos\mathrm{H}\cos\zeta \\ &\quad - \cos(\Phi-\psi)\sin\mathrm{H}\sin\epsilon\}]. \end{aligned} \tag{16}$$

We can write equation (16),

$$V_R(\phi,\eta) = F_1\mu_1 + F_2\mu_2 + F_3\mu_3 \tag{17}$$

where

$$F_1 = n_2 r_2 \omega\{\sin\epsilon\sin(\Phi-\psi)\sin \mathrm{H}\sin\zeta + \cos \mathrm{H}\cos\zeta\sin\epsilon\}$$
$$F_2 = n_2 r_2 \omega\{\cos\epsilon\sin(\Phi-\psi)\sin \mathrm{H}\sin\zeta + \cos\epsilon\cos \mathrm{H}\cos\zeta\}$$
$$F_3 = n_2 r_2 \omega\{-\cos\epsilon\sin(\Phi-\psi)\sin \mathrm{H}\cos\zeta + \cos\epsilon\cos \mathrm{H}\cos\zeta - \cos(\Phi-\psi)\sin \mathrm{H}\sin\epsilon\}. \tag{18}$$

All the functions of equation (1) are now known, and the solution of the problem now consists only of applying the K-operators to the product $V_R(\phi,\eta)\,\mathscr{I}(\phi,\eta)$. This in turn reduces to applying the operators to terms of the form $\mu_1^p (1 \leqslant p \leqslant 5)$, $\mu_1^p\mu_2$ and $\mu_1^p\mu_3 (0 \leqslant p \leqslant 4)$ where p is an integer. This is done in the next section.

First, however, let us define

$$l_0 = \cos\psi\sin i = \cos\epsilon$$
$$m_0 = \sin\psi\sin i$$
$$n_0 = \cos i.$$

Then, introducing these, together with equations (13) into equations (18) we obtain

$$F_1 = r_2(\omega_y' n_0 + \omega_z' m_0)$$
$$F_2 = r_2(\omega_y' n_0 + \omega_z' m_0)\cot\epsilon$$
$$F_3 = r_2\{(\omega_z' - \omega_y')\, m_0 - \omega_x'\sin\epsilon\tan\epsilon\}\cot\epsilon. \tag{19}$$

4. *The K-integrals of μ_1^p, $\mu_1^p\mu_2$ and $\mu_1^p\mu_3$.*—The integrals $K(\sin\eta_1)\{\mu_1^p\}$ and $\{K(\sin\eta_2) - K(\sin\eta_1)\}\{\mu_1^p\}$ have been determined for some of the smaller values of p, and the results have been used in the summarized data of section 2. Kopal showed in (3) that the integrals $K(\sin\eta_1)\{\mu_1^p\}$ could be represented entirely by four functions of ϵ and η_1 (or η_2 for $K(\sin\eta_2)\{\mu_1^p\}$) for $0 \leqslant p \leqslant 3$ and the present writer has verified that this remains true for $p = 4$ and 5. The four functions are

$$\Phi_1 = \cos^{-1}(\sin\eta_1\operatorname{cosec}\epsilon)$$
$$= \frac{\pi}{2} - \sin\eta_1\operatorname{cosec}\epsilon - \frac{1}{6}\sin^3\eta_1\operatorname{cosec}^3\epsilon - \frac{3}{40}\sin^5\eta_1\operatorname{cosec}^5\epsilon\ldots$$
$$\Phi_2 = \cos^{-1}(-\sec\eta_1\cos\epsilon)$$
$$= \pi - \epsilon + \frac{1}{2}\cot\epsilon\sin^2\eta_1 + \frac{1}{8}\sin^4\eta_1(\cot^2\epsilon + 3)\cot\epsilon$$
$$+ \frac{1}{48}\sin^6\eta_1\cot\epsilon(15 + 10\cot^2\epsilon + 3\cot^4\epsilon)\ldots$$
$$\Phi_3 = \cos^{-1}(\tan\eta_1\cot\epsilon)$$
$$= \frac{\pi}{2}\tan\eta_1\cot\epsilon + \frac{1}{6}\tan^3\eta_1\cot^3\epsilon + \frac{3}{40}\tan^4\eta_1\cot^5\epsilon\ldots$$
$$F = (\sin^2\epsilon - \sin^2\eta_1)^{1/2}$$
$$= \sin\epsilon - \frac{1}{2}\sin^2\eta_1\operatorname{cosec}\epsilon - \frac{1}{8}\sin^4\eta_1\operatorname{cosec}^3\epsilon - \frac{1}{16}\sin^6\eta_1\operatorname{cosec}^5\epsilon\ldots. \tag{20}$$

and the forms of $K(\sin\eta_1)\{\mu_1^p\}$ for $0 \leqslant p \leqslant 5$ are given by:

$$K(\sin\eta_1)\{\mu_1^0\} = r_2^2\{\Phi_1 + \cos\epsilon\cos^2\eta_1\Phi_3 - \sin\eta_1 F\}$$

$$K(\sin\eta_4)\{\mu_1^1\} = \frac{2}{3}r_2^2\cos\epsilon(\Phi_2 - \Phi_3\sin^3\eta_1) + \frac{2}{3}r_2^2\cos^2\eta_1 F$$

$$K(\sin\eta_1)\{\mu_1^2\} = \frac{r_2^2}{4}(1+\cos^2\epsilon)\,\Phi_1 + \frac{r_2^2}{2}(1-\sin^4\eta_1)\cos\epsilon\,\Phi_3 + \frac{r_2^2}{4}\sin\eta_1(1-2\sin^2\eta_1)F$$

$$K(\sin\eta_1)\{\mu_1^3\} = \frac{2}{5}r_2^2\cos\epsilon\,(\Phi_2-\Phi_3\sin^5\eta_1) + \frac{2}{5}r_2^2(1-\sin^4\eta_1)F - \frac{2}{15}r_2^2F^3$$

$$K(\sin\eta_1)\{\mu_1^4\} = \frac{r_2^2}{24}(3+\cos^2\epsilon\,(\sin^2\epsilon+5))\,\Phi_1 + \frac{r_2^2}{3}(1-\sin^6\eta_1)\cos\epsilon\,\Phi_3$$
$$+\frac{r_2^2\sin\eta_1}{24}(4+\sin^2\eta_1-8\sin^4\eta_1)\,F - \frac{r_2^2}{24}\sin\eta_1\,F^3$$

$$K(\sin\eta_1)\{\mu_1^5\} = \frac{2r_2^2}{7}\cos\epsilon\,(\Phi_2-\sin^7\eta_1\,\Phi_3) + \frac{2r_2^2}{7}(1-\sin^6\eta_1)\,F$$
$$-\frac{2r_2^2}{21}(1+\sin^2\eta_1)\,F^3 - \frac{4r_2^2}{105}F^5. \tag{21}$$

The form of the integrals $K(\sin\eta_1)\{\mu_1^p\mu_2\}$ may be obtained by substituting in equation (9) $\cos^p\phi\,\sin\phi$ for $\cos^p\phi$. The power of $\sin\eta$ in the first integral will remain unchanged. It has been found that the same four functions of equations (20) may be used to express these integrals as well, and the results for $0 \leqslant p \leqslant 4$ are quoted in equations (22).

$$K(\sin\eta_1)\{\mu_1^0\mu_2\} = r_2^2\left\{\frac{2}{3}\sin\epsilon\,\Phi_2 + \frac{\sin\epsilon\sin\eta_1}{3}(\sin^2\eta_1-3)\,\Phi_3 - \frac{1}{3}\cot\epsilon\sin^2\eta_1\,F\right\}$$

$$K(\sin\eta_1)\{\mu_1^1\mu_2\} = r_2^2\left\{\frac{\sin\epsilon\cos\epsilon}{4}\Phi_1 + \frac{\sin\epsilon}{4}(\sin^4\eta_1-2\sin^2\eta_1+1)\,\Phi_3\right.$$
$$\left.-\frac{1}{4}\sin^3\eta_1\cot\epsilon\,F\right\}$$

$$K(\sin\eta_1)\{\mu_1^2\mu_2\} = \frac{2}{15}r_2^2\sin\epsilon\,\Phi_2 + \frac{r_2^2}{15}\sin\epsilon\sin^3\eta_1(3\sin^2\eta_1-5)\,\Phi_3$$
$$-r_2^2\left(\frac{1}{5}\sin^4\eta_1\cot\epsilon - \frac{2}{15}\sin\epsilon\cos\epsilon\right)F$$

$$K(\sin\eta_1)\{\mu_1^3\mu_2\} = r_2^2\frac{\sin\epsilon\cos\epsilon}{24}(\sin^2\epsilon+2)\,\Phi_1 + \frac{r_2^2\sin\epsilon}{12}(2\sin^6\eta_1-3\sin^4\eta_1+1)\,\Phi_3$$
$$-\frac{r_2^2}{6}(\sin^5\eta_1\cot\epsilon - \frac{1}{4}\sin\eta_1\sin\epsilon\cos\epsilon)\,F$$

$$K(\sin\eta_1)\{\mu_1^4\mu_2\} = \frac{2}{35}r_2^2\sin\epsilon\,\Phi_2 + \frac{r_2^2\sin\epsilon\sin^5\eta_1}{35}(5\sin^2\eta_1-7) - \frac{r_2^2}{7}\sin^6\eta_1\cot\epsilon\,F$$
$$+\frac{2r_2^2}{35}\sin\epsilon\cos\epsilon\,(1+\sin^2\eta_1)\,F + \frac{4}{105}r_2^2\sin\epsilon\cos\epsilon\,F^3. \tag{22}$$

Finally if in equation (9) we replace $\sin^{p+2}\eta$ by $\sin^{p+1}\eta\cos\eta$ we get the form of the integrals $K(\sin\eta_1)\{\mu_1^p\mu_3\}$. It is then immediately seen that the first integration yields $(\sin^{p+2}\eta)/(p+2)$, which taken between the limits of the integration vanishes identically. We may, therefore, omit from all further consideration the term $F_3\mu_3$ in equation (17).

The integrals $K(\sin\eta_2)$ are, of course, obtained simply by substituting $\sin\eta_2$ for $\sin\eta_1$ in equations (20), (21), (22). Again we have to apply the operator $K(\sin\eta_2)-K(\sin\eta_1)$ to two types of integrand which arise from the multiplication

of equations (18) and (4). The term $F_1\mu_1$ in equation (18) produces the following integrand

$$\frac{r_1 L_1 F_1}{\pi R^3} \sum_{n=0}^{4} C_n^{U,D} \left(\frac{R}{r_1}\right)^n \mu_1^{n-1} \tag{23}$$

while the term $F_2\mu_2$ produces the integrand

$$\frac{r_1 L_1 F_2}{\pi R^3} \sum_{n=0}^{4} C_n^{U,D} \left(\frac{R}{r_1}\mu_1\right)^n \mu_2. \tag{24}$$

Now Kopal has shown in (3), and it may be readily verified from the data given here, that

$$\{K(\sin\eta_2) - K(\sin\eta_1)\}\left\{\frac{r_1 L_1}{\pi R^3} \sum_{n=0}^{4} C_n^{U,D} \left(\frac{R}{r_1}\mu_1\right)^n\right\}$$
$$= \frac{2}{\pi}\left(\frac{r_1 r_2}{R^2}\right)^2 L_1 \sin\epsilon \sum_{n=0}^{4} \frac{C_n^{U,D}}{n+1}\left\{\left(\frac{r_1+r_2}{r_2}\right)^{n+1} - \left(\frac{r_2-r_1}{r_2}\right)^{n+1}\right\}$$

(to the order of accuracy adopted) so that it follows that the integrand (23) will be of the fifth order in r/R, and consequently is negligible within the limits of our approximation. Similarly, as the reader may verify, we have

$$\{K(\sin\eta_2) - K(\sin\eta_1)\}\{\mu_1^n \mu_2\} \simeq \frac{r_2^2}{n+1}\,\frac{\pi}{2}(\sin^{n+1}\eta_1 - \sin^{n+1}\eta_2)\sin\epsilon \tag{25}$$

Taking into account the dependence of the coefficients of (24) on the ratio r_1/R it is seen that equation (25) is of the order $(r/R)^4$ for all n.

We are now in a position to write down the full expression for the integral in the numerator of equation (1). Let us denote its value in the full light zone by I_F and in the penumbral zone by I_p so that

$$\int_{\Sigma(\epsilon)} V_R(\phi,\eta)\,\mathscr{I}(\phi,\eta)\,d\sigma = I_F + I_p$$

where

$$I_F = F_1 L_1\left\{\frac{1}{8}\left(\frac{r_2}{R}\right)^2(1+\cos\epsilon)^2 + \frac{2}{15\pi}\left(\frac{r_2}{R}\right)^3(4[(\pi-\epsilon)\cos\epsilon + \sin\epsilon] - 3\sin^3\epsilon)\right.$$
$$\left. - \frac{1}{32}\left(\frac{r_2}{R}\right)^4(5\cos^4\epsilon - 12\cos^2\epsilon - 4\cos\epsilon + 3)\right\}$$
$$+ F_2 L_1\left\{\frac{1}{8}\left(\frac{r_2}{R}\right)^2(1+\cos\epsilon)\sin\epsilon - \frac{2}{15\pi}\left(\frac{r_2}{R}\right)^3(4[\pi-\epsilon]\sin\epsilon - 3\sin^2\epsilon\cos\epsilon)\right.$$
$$\left. + \frac{\sin\epsilon(1+\cos\epsilon)}{32}\left(\frac{r_2}{R}\right)^4(5\cos^2\epsilon - 5\cos\epsilon - 8) - \frac{\pi}{4}\frac{r_2^2 r_1^2}{R^4}\sin\epsilon\right\} \tag{26}$$

and

$$I_p \simeq \frac{F_2 L_1 \sin\epsilon}{10\pi(3-u_1)}\,\frac{r_1^2 r_2^2}{R^4}\{30(1-u_1) + 6\pi u_1\}. \tag{27}$$

The approximation in equation (27) arises from the neglect of one small term in the summation and involves an error of $\sim$0·5 per cent. This is, in principle, the solution of the problem; there is, however, one further refinement to consider before applying the formula.

5. *Secondary reflection.*—It has been pointed out that a reflection law which includes terms of the form $r_1^2 r_2^2/R^4$ must, to be consistent, include the effect of secondary reflection; that is the effect of light which has been reflected from one star back to the other, and thence to the observer. It is clear that this phenomenon effectively increases the luminosity of the illuminating star. Kopal has shown in

(3) that the effect of the increase is that L_1 in his formula for $L_R(\epsilon)$ (equation 10) should be replaced by $L_1 + \frac{2}{3}r_1^2 L_2/R^2$. It is obvious that, within our approximation, this is equivalent to adding an extra term in $r_1^2 r_2^2/R^4$ to the right hand side of equation (10). Since reflected light obeys a different law of limb darkening from that of the star's proper light, one might expect the distribution of intensity over the surface of the illuminating star to be changed, with a consequent modification of $\mathscr{I}(\phi, \eta)$ in the penumbral zone. Such an effect, however, would be small since the restriction we have made that both stars be spherical in fact amounts to an upper limit of r/R of $\sim 0{\cdot}1$ or $0{\cdot}2$ so that reflected light is never more than a few per cent of the star's proper light, and its effect on the distribution of intensity can be neglected. Nonetheless, this neglect represents a further danger in extrapolating our results to very close binary systems or to systems in which one component is so dark that radiation reflected from it becomes comparable in intensity to that of the star itself.

6. *The correction to the masses of binary components.* By substituting the results of equations (10), (26) and (27) into equation (1), and remembering to write $L_1 + \frac{2}{3}r_1^2 L_2/R_2$ for L_1 we can obtain the explicit form of the correction $\Delta v(\epsilon)$ in terms of the luminosities and radii of the two components together with the rotational velocity of the reflecting component. This formula, if written out explicitly, could, in principle, be used to rectify a velocity curve before a solution was made from it for the elements of the system. If, further, the notation of equation (19) is used then the phase law of the velocity variation is given by

$$\Delta V(\epsilon) = \frac{r_2(\omega_y' n_0 + \omega_z' m_0)}{L_2 + L_R(\epsilon)} L_1 \left\{ \frac{1}{8}\left(\frac{r_2}{R}\right)^2 (1+\cos\epsilon) + \frac{2}{15\pi}\left(\frac{r_2}{R}\right)^3 [6(\pi-\epsilon)\cos\epsilon + \sin\epsilon] + \frac{3}{32}\left(\frac{r_2}{R}\right)^4 (5\cos\epsilon - 1)(1+\cos\epsilon) + \frac{\pi}{4}\frac{r_1^2 r_2^2}{R^4}\cos\epsilon + \frac{1}{12}\frac{L_2}{L_1}\frac{r_1^2 r_2^2}{R^4}(1+\cos\epsilon) \right\}. \tag{28}$$

To obtain the correction Δm to the mass of the reflecting component we require only to know

$$\Delta v\left(\frac{\pi}{2}\right) = \Delta K_2$$

We have at once that

$$\cos\epsilon = \cos\psi = 0, \quad \sin\epsilon = \sin\psi = 1$$

and so

$$m_0 = \sin i.$$

The minus sign in front of the right hand side of equation (1) can be ignored since we are no longer interested in the direction of the radial velocity, but only in the absolute value of the semi-amplitude of the velocity curve. This is always increased by the correction ΔK_2 unless $\cos H < 0$ and $n_2 \geqslant 1$. This exception corresponds to retrograde rotation, and for $\cos H = -1$, $n_2 = 1$ we have $\Delta K_2 = 0$. However, retrograde rotation is an unknown phenomenon in binary systems, and the change of sign in F_1 would, in any case, ensure that the correction ΔK is always applied in the right sense. The general formula, therefore, reduces to

$$\Delta K_2 = \frac{F_1 \left\{ \frac{1}{8}\left(\frac{r_2}{R}\right)^2 + \frac{2}{15\pi}\left(\frac{r_2}{R}\right)^3 - \frac{3}{32}\left(\frac{r_2}{R}\right)^4 + \frac{1}{12}\frac{r_1^2 r_2^2}{R^4}\frac{L_2}{L_1} \right\}}{(L_2/L_1) + \left\{ \frac{2}{3\pi}\left(\frac{r_2}{R}\right)^2 - \frac{1}{8}\left(\frac{r_2}{R}\right)^3 + \left(\frac{12}{5\pi^2}\cdot\frac{5+(\pi-5)u_1}{3-u_1} - \frac{1}{\pi} + \frac{4}{9\pi}\frac{L_2}{L_1} \right)\frac{r_1^2 r_2^2}{R^4} \right\}} \tag{29}$$

where F_1 is defined by equation (19) and L_2/L_1 by

$$\frac{L_2}{L_1}=\frac{r_2^2 T_2^4}{r_1^2 T_1^4} \tag{30}$$

where $T_{1,2}$ are the effective temperatures of the two stars. The suffix 2 has been used to denote the reflecting star, so that ΔK for the other star is obtained simply by exchanging the suffixes 1 and 2 in equation (29) (except for F_1 which is adjusted by substituting in it n_1 for n_2). If $r_{2,1}$ are expressed in kms and ω in radians/sec. $\Delta K_{1,2}$ are obtained in km/sec in equation (29). The ΔK's must be divided by the factor $\sin i$ to obtain the true velocities of the system. If then P is the period of the system in days, the true value of R in kms is given by

$$R\sin i = 13{,}750P(K_1+K_2+\Delta K_1+\Delta K_2). \tag{31}$$

The correction ΔR to be applied to any elements already deduced is

$$\Delta R\sin i = 13{,}750P(\Delta K_1+\Delta K_2)$$

($K_{1,2}$ are here supposed to be the *observed* semi-amplitudes of the velocity curves, and are, therefore, the true values multiplied by $\sin i$).

If R is then expressed in astronomical units and P in years and m in solar masses, then,

$$\begin{aligned}(m_1+m_2+\Delta m)\sin^3 i &= R^3/P^2\\ &= 1{\cdot}036\times 10^{-7}P(K_1+K_2+\Delta K_1+\Delta K_2)^3\end{aligned}$$

if R and P are to be expressed in the units of equation (31) but m still in solar masses, and

$$\Delta m\sin^3 i = 1{\cdot}036\times 10^{-7}P[3(K_1+K_2)^2(\Delta K_1+\Delta K_2)+3(K_1+K_2)(\Delta K_1+\Delta K_2)^2] \tag{32}$$

since the term $(\Delta K_1+\Delta K_2)^3$ is of the order $(r/R)^6$, provided L_2/L_1 is not too small.

Also

$$\frac{m_1+\Delta m_1}{m_2+\Delta m_2}=\frac{K_2+\Delta K_2}{K_1+\Delta K_1}. \tag{33}$$

Equations (32) and (33) give the corrections to be made to m_1 and m_2 since

$$\Delta m = \Delta m_1 + \Delta m_2.$$

In one-spectrum systems the corrections to the mass function is

$$\Delta\frac{m_1^3\sin^3 i}{(m_1+m_2)^2} = 1{\cdot}036\times 10^{-7}\,P(3K_2^2\Delta K_2+3(\Delta K_2)^2K_2). \tag{34}$$

7. ***The magnitude of the effect.***—As has already been stated, the results of the previous section are only rigorously applicable to binary systems in which both components are sensibly spherical in shape. This implies $r/R \lesssim 0{\cdot}2$ as otherwise tidal effects would begin to be significant. In such systems there is no constraint on either star to rotate with any particular speed or about any particular axis. Furthermore, even if we succeed in detecting rotation of the components of these systems we have no means of deriving either the speed or the inclination of the axis, but only a combination of the two. The function F_1 is, therefore, indeterminate. In closer systems the tidal bulge will tend to pull the axis of rotation perpendicular to the orbital plane ($H=0$) and also to make the period of rotation coincide with that of revolution ($n_2=1$). In these systems, however, the assumption of sphericity breaks down. The formulae derived, therefore, are of only limited practical use until the effects of reflection from distorted stellar surfaces are considered—a task it is hoped to undertake soon. Nonetheless, it may be useful

to employ equation (30) to estimate the order of magnitude of the maximum possible effect in a system in which the components rotate with the Keplerian angular velocity about an axis perpendicular to the orbital plane ($F_1 = r_2\omega \sin i$). The maximum effect is observed when the reflecting star is completely dark. Then the error, Δm refers to the illuminating star only, for the illuminating star receives no light to reflect. Equation (30) reduces to

$$\Delta K_2 = \frac{r_2\omega \sin i\left\{\frac{1}{8}\left(\frac{r_2}{R}\right)^2 + \frac{2}{15\pi}\left(\frac{r_2}{R}\right)^3 - \frac{3}{32}\left(\frac{r_2}{R}\right)^4\right\}}{\frac{2}{3\pi}\left(\frac{r_2}{R}\right)^2 - \frac{1}{8}\left(\frac{r_2}{R}\right)^3 + \left(\frac{12}{5\pi^2}\frac{5+(\pi-5)u_1}{3-u_1} - \frac{1}{\pi}\right)\frac{r_1^2 r_2^2}{R^4}}$$

while

$$K_2 = \frac{m_1}{m_1+m_2} R\omega \sin i.$$

In such a system, m_1 is probably several times larger than m_2 and, since we are interested only in orders of magnitude, we may ignore the factor $m_1/(m_1+m_2)$. If we also set $u_1 = 0{\cdot}5$, then

$$\frac{\Delta K_2}{K_2} = \frac{r_2}{R}\left\{0{\cdot}5890 + 0{\cdot}5470\frac{r_2}{R} - 0{\cdot}1197\left(\frac{r_2}{R}\right)^2 - 0{\cdot}0704\left(\frac{r_2}{R}\right)^3 - 0{\cdot}2155\left(\frac{r_1}{R}\right)^2 - 0{\cdot}3272\left(\frac{r_2 r_1}{R^3}\right)^2\right\}$$

while

$$\frac{m_1+\Delta m_1}{m_1} = \frac{(K_2+\Delta K_2)^3}{K_2^3}$$

where m_1 is the *observed* mass of the primary component, whence

$$\frac{\Delta m_1}{m_1} = \frac{3\Delta K_2}{K_2} + 3\left(\frac{\Delta K_2}{K_2}\right)^2 + \left(\frac{\Delta K_2}{K_2}\right)^3$$

which to our order of accuracy gives

$$\frac{\Delta m_1}{m_1} = +1{\cdot}7670\frac{r_2}{R} + 2{\cdot}6818\left(\frac{r_2}{R}\right)^2 + 1{\cdot}7783\left(\frac{r_2}{R}\right)^3 + 0{\cdot}8354\left(\frac{r_2}{R}\right)^4 - 0{\cdot}6465\frac{r_1^2 r_2}{R^3} - 1{\cdot}7345\frac{r_1^2 r_2^2}{R^4} \qquad (35)$$

From this we see that for $r_2/R = 0{\cdot}1$ the maximum correction which may be made to the mass of the system is about one fifth of the observed mass of the heavier component. For $r_2/R \simeq 0{\cdot}3$ the formula (35) certainly gives a value of the same order as the observed mass itself, as suggested by Ovenden, but it takes no account of the distortion which must most certainly exist in such a system. If the distortion of the reflecting star is small enough for the star to be regarded as an ellipsoid, then, although the precise corrections to equation (35) cannot yet be evaluated, they are unlikely to affect the order of magnitude of the result. However, many systems in which $r_2/R \sim 0{\cdot}3$ are "semi-detached" systems in which the reflecting star is the contact component. As a result, gravity darkening will cause the distribution of the star's proper light to be asymmetric in the opposite sense to the distribution of light it reflects, and so equation (35) would break down completely and give a gross over-estimation of $\Delta m_1/m_1$.

By contrast we can also consider the case $L_1 = L_2$ which will give the minimum correction for any given value of r/R—this time to be applied to both stars. In this case we shall also take $m_1 = m_2$ so that

$$K_{1,2} = \tfrac{1}{2} R\omega \sin i.$$

Again we take $u_1 = 0{\cdot}5$; $\left(\frac{\Delta K}{K}\right)^2$ and $\left(\frac{\Delta K}{K}\right)^3$ are now negligible quantities to the order of accuracy adopted, so that

$$\frac{\Delta m_{1,2}}{m_{1,2}} = \frac{3\Delta K_{1,2}}{K_{1,2}}$$

and

$$\frac{\Delta m_1}{m_1} = 0{\cdot}75\left(\frac{r_2}{R}\right)^3 + 0{\cdot}2547\left(\frac{r_2}{R}\right)^4. \tag{36}$$

For two stars of equal mass and luminosity, the radii may also be taken as equal, so that to obtain the total correction to the mass of the system equation (36) should be multiplied by 2.

Substituting in equation (36) $r_2/R = 0{\cdot}1$ we find the correction to the mass to be quite negligible, while for $r_2/R = 0{\cdot}3$ it will be 2 per cent of the observed mass of one component, or a correction of about 4 per cent of the observed mass of one component to the whole system. It should be emphasized that equations (35) and (36) are only true if the component stars are rotating with the Keplerian angular velocity about an axis perpendicular to the orbital plane. Unless, however, the angular velocity of rotation greatly exceeds that of revolution, both equations take a reasonable account of the effect of rotation.

Summarizing, we may say that in certain cases corrections, due to reflection, to observed masses of the same order as the masses themselves may occasionally be expected, but that such corrections arise in circumstances where theory developed in this paper breaks down, so that their precise magnitude may not yet be estimated. On the other hand, in systems with $r/R \sim 0{\cdot}1$ and in which the two components do not differ greatly in luminosity no correction to the masses for the reflection effect need be applied. For the general run of systems in which $0{\cdot}1 \lesssim r/R \lesssim 0{\cdot}2$ and in which the components are reasonably similar in mass and luminosity corrections to the total mass of the system of the order of 10 per cent of the mass of the heavier component may be expected. This last conclusion confirms that reached by Kopal in (4) where the illuminating star was considered as a point source.

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FUNDAMENTAL DATA FOR SOUTHERN STARS (FIRST LIST)

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Summary

Newly determined radial velocities are given for 25 IAU standard velocity stars and 343 other stars south of -26°. Many nearby stars and stars of high velocity are included in this list. Magnitudes and colours on the Johnson *B*, *V* system and spectral types on the MK system have been determined for all the stars in the main list. Diagrams showing the relation between absolute magnitude, spectral type and colour index are given for the stars with known trigonometrical parallaxes and from these are deduced photometric parallaxes for 142 stars of luminosity class V. Space coordinates and space motions have been computed for these stars.

1. *Introduction.*—In 1951, as the result of an agreement between the Admiralty and the Radcliffe Trustees, one third of the observing time with the 74-inch Radcliffe reflector was made available to the Cape Observatory. It was mutually agreed that a considerable portion of this time should be devoted to the determination of the radial velocities of the later type stars while the Radcliffe observers were to concentrate on early type stars and special objects. The first Cape programme, begun in 1951, included about 400 stars south of declination -26° chosen either for their large parallaxes, or their large proper motions, or because they lie near one of the Kapteyn Selected Areas, Nos. 140–206, and have annual proper motions exceeding $0''.1$. To this initial programme there were later added a number of visual double stars suggested by Dr W. H. van den Bos as likely to be worth observing, a short list of interferometric double stars selected by Dr W. S. Finsen, and a list of 25 stars, which are, or have been, IAU standard velocity stars. These latter were included as a result of the request by Dr J. A. Pearce, Chairman of the Sub-Commission on Radial Velocity Standards, that the equatorial and southern IAU standards be reobserved. As the observations proceeded, certain parts of the programme were worked out more rapidly than others and were filled in by selecting additional stars from the appropriate Selected Areas.

The present list gives the results for those stars for which the radial velocity observations had been completed before 1956 June 1. Spectral types on the MK system have been estimated from the radial velocity plates and accurate magnitudes and colours for all the stars have been measured with the photoelectric photometers at the Cape Observatory. As many of the stars have well-determined trigonometrical parallaxes it has been possible to make a new calibration of the Hertzsprung–Russell main sequence for the later type stars.

2. *The radial velocity observations.*—An account of the Radcliffe Cassegrain spectrograph and of its performance has been given by Feast, Thackeray and Wesselink (1). The three cameras that have been used for the Cape programme are those known as "a", "b" and "c" which give dispersions of 21, 29 and 49 A/mm at Hγ. Neutral filters having absorptions of 1·3, 2·6 or 5·2 magnitudes have been

placed before the slit when observing bright stars in order to obtain an exposure time of at least ten minutes so that any errors produced by seeing variations should be properly smoothed out.

The spectra have been measured with standard Hilger long screw machines, either with that belonging to the Radcliffe Observatory or with one or other of the two at the Cape Observatory. Most of those who have taken part in the observations have measured some plates to gain experience but, because of the appreciable personal equations that have been detected, only the measures by Dr David S. Evans (DSE) and by Mr A. Menzies (AM) have been retained. The selection of the lines to be measured and their adopted normal wave-lengths were taken from various publications of the Dominion Astrophysical Observatory ; from *Con. D. A. O.* Nos. 4 and 10 for the "a" and "b" camera plates, from *Con. D. A. O.* Nos. 11 and 12 for the "c" camera plates. Later, when *Publ. D. A. O.*, **9**, No. 3 became available, the selection of lines given in it was used for reducing the spectra of F4–M8 stars observed with the "b" camera. The Victoria standards have proved themselves reliable and self-consistent in every way.

All the data available at 1955 June 1 were used to derive the systematic differences between the two measurers and the corrections needed to reduce their measures to the system of the Mt Wilson General Catalogue of Radial Velocities. At the same time eleven conveniently placed "reference" stars were chosen. Seven of these are IAU standards and are indicated by an "R" number in Table I, while the remaining four are given at the end of Table II. These stars are being observed frequently, usually one such star per full night of observation, and each of their spectra is being measured by both of the measurers in order to keep a continuous control on the necessary systematic corrections. Following a suggestion by Feast, Thackeray and Wesselink, the radial velocities of the brighter minor planets are being observed in an attempt to evaluate the correction required by the observed system of radial velocities to reduce it to an absolute one.

The systematic corrections in km/s that have actually been applied to the measures are as follows :—

	"a"		"b"		"c"	
DSE	+1·7	...	+0·7	+0·5	−1·2	−0·9
AM	+1·7	...	+0·2	+0·7	−2·5	−0·5

The first set of figures refers to plates taken before 1955 June 1 and the second to plates taken during the year between 1955 June 1 and 1956 June 1. The "a" dispersion was not used during this latter period. These corrections correspond to a linear displacement on the plate of a micron or less.

The velocities of the 25 IAU standard stars given in Table I were determined in 1954 and 1955 from a minimum of six plates taken with either the "a" or the "b" dispersion. Each plate was measured by both measurers and these measures were combined after the appropriate systematic corrections had been applied. Probable errors of the finally adopted velocities were derived both from the probable errors of the individual plates and from the interagreement of the separate measures. In every case the probable error of the adopted velocity derived by the first method is ±0·2 km/s. The probable error estimated from the interagreement of the individual plates is ±0·4 km/s, or less, for 22 of the stars, the actual average being ±0·25 km/s. For HD 182572 it is ±0·8 km/s and for

HD 119971 and HD 223311 ±0·5 km/s, so that the velocities of these stars are probably variable. If, omitting these stars, the double star HD 51250 and the star HD 173009 for which there is an unduly large and unexplained difference, we compare the new velocities with those of *a* quality given in the Mt Wilson Catalogue we find that for 17 stars the average difference between the two sets of velocities is 0·0 km/s. This is the result to be expected since the systematic corrections to the Cape measures were derived principally from a consideration of these stars. A more comprehensive comparison with the Mt Wilson Catalogue is given later.

The measured radial velocities of the programme stars and their probable errors are given in the eighth column on Table II. Each depends on a minimum of four plates and usually rather more. The probable errors are the mean of those deduced from the interagreement of the plates and those deduced from the formal probable errors of the individual plates. The cases in which the first estimate exceeds the second sufficiently to justify suspicion of variability in the velocity are indicated in the Notes. The last column of Table II gives the actual number of plates measured for each star.

A fairly large number of stars in Table II appears in the Mt Wilson Catalogue. Some were included in the first selection of stars to be observed in order to give a strong link between the new velocities and previous work. Using all the stars in Tables I and II for which a radial velocity of *a* or *b* quality is given in the Mt Wilson Catalogue, and arranging them in the usual groups according to spectral types, we get as the systematic differences between the observed velocities and the values given in the Catalogue the following :

Sp. Qual.	B	A	F	G	K	M
a		+1·2 (1)	−1·7 (3)	0·0 (23)	+0·1 (31)	+0·7 (8)
b	+2·8 (1)	+0·5 (6)	−1·2 (6)	−1·6 (9)	−1·4 (16)	−1·0 (4)
a+*b*	+2·8 (1)	+0·6 (7)	−1·4 (9)	−0·4 (32)	−0·4 (47)	+0·1 (12)

the differences being in km/s and in the sense Mt Wilson Catalogue-Cape.

In whatever way the data are handled, they produce a feeling of some confidence that any differences that may exist between the observed system of these new velocities and the system of the Mt Wilson Catalogue do not exceed the probable errors of the material under discussion.

3. *The photometric observations.*—All the stars being observed for radial velocity have been included in a working list of about 4000 bright southern stars for which the magnitudes and colours are being measured with a photoelectric photometer attached to the 13-inch Cape Astrographic Refractor. The normal routine of measurement at the telescope has been blue sky, blue star, yellow star, yellow sky, yellow star, blue star, blue sky, the individual exposures being of approximately 30 seconds duration. The net deflexions (star–sky) are read off from the Brown Recording Potentiometer chart directly in magnitudes to the nearest $0^m{\cdot}01$ by means of a perspex scale specially engraved for the purpose. The first, last and every third intermediate star observed is a standard. The working list is so arranged that all stars between -4° and -64° declination are observed at 30° zenith distance, while the relatively few stars in the programme outside these limits have been observed near culmination and compared with E region standard stars at equal altitudes. Where necessary, differential extinction corrections have been applied, the assumed zenithal extinctions being $0^m{\cdot}35$ for

the blue and $0^{m}{\cdot}20$ for the yellow, which correspond to average photometric conditions at the Cape. In actual practice, extinction corrections have to be applied to relatively few observations and very rarely exceed $0^{m}{\cdot}01$. Each star receives a minimum of four observations on separate nights and the probable errors of the resulting magnitudes and colours as deduced from the internal agreement are $\pm 0^{m}{\cdot}005$ and $\pm 0^{m}{\cdot}004$ respectively.

The stars that are being used as standards for the "30°" portion of the photometric programme are those observed in the Cape "Bright Star Programme", the results of which are given in *Cape Mimeogram No.* 1 (1953). The Fabry magnitudes (BPg) and the photoelectric colours (C_{pe}) given there were approximately transformed into the natural colour system (b, c) of the Astrographic observations by means of the relations

$$b = \mathrm{BPg} - 0{\cdot}18\ C_{pe}$$
$$b - y = c = 1{\cdot}11\ C_{pe} - 0{\cdot}35$$

after the following small zero point corrections, which resulted from a special series of observations, had been applied :

$+0^{m}{\cdot}01$ to the magnitudes of all stars between HR 1298 and HR 2451
and between HR 4441 and HR 5068
to the colours of all stars between HR 7063 and HR 9098
$-0^{m}{\cdot}01$ to the magnitudes of all stars between HR 3165 and HR 3940
and between HR 7063 and HR 8204
to the colours of all stars between HR 3165 and HR 4216

An examination of the results of the first three years' observations for this programme showed that the magnitudes and colours that had been thus derived for the standard stars were essentially reliable but that further small corrections, which could be partially explained as due to a "bandwidth" effect, were necessary. These were as follows :

c	Δb	c	Δc
m	m	m	m
	−0·01		−0·01
−0·50		−0·50	
	0·00		0·00
−0·25		+0·50	
	+0·01		+0·01
+0·65			
	0·00		
+0·90			
	−0·01		

The relations used for transforming the observed Astrographic visual magnitudes and colours into the Johnson V, $B-V$ system are

$$V = y - 0{\cdot}39$$
$$B - V = 0{\cdot}92\,c + 0{\cdot}52$$

These were derived by a direct comparison of the magnitudes and colours of 30 stars as observed with the Astrographic with those given by Johnson in *Annales d' Astrophysique* (**2**) which, for several of the stars in question, differ appreciably

from those given in the original paper of Johnson and Morgan (3). The average differences between the Johnson and Cape measures are $0^{m}\cdot017$ in magnitude and $0^{m}\cdot012$ in colour, which are rather larger than might be expected from the quality of the data involved.

The magnitudes and colours given in the fifth and sixth columns of Table II are, with very few exceptions, derived from recent observations with the Astrographic Refractor. For a few stars brighter than $5^{m}\cdot0$ and south of -64°, the magnitudes and colours have been deduced from the earlier observations given in *Cape Mimeogram No.* 1 according to the precepts given above. For the bright stars north of -64°, which have been used as standards for the normal "30°" programme, the magnitudes and colours given are those deduced from the actual Astrographic observations.

The diaphragm of the Astrographic photometer has a diameter of $1'\cdot5$ so that it has not been possible to measure individual magnitudes and colours for a number of stars that are too close together to be effectively separated. Stars for which the magnitudes and colours fail for any reason to come up to the usual standard of accuracy are indicated in Table II by : and are referred to in the Notes.

4. *The spectral types.*—The spectral types and luminosity classes given in the seventh column of Table II have been estimated from the radial velocity plates by Dr David S. Evans. They are intended to be on the MK system. The types were first estimated using the standards and criteria of the MKK atlas (4) and then the modifications set out by Johnson and Morgan (3) were applied.

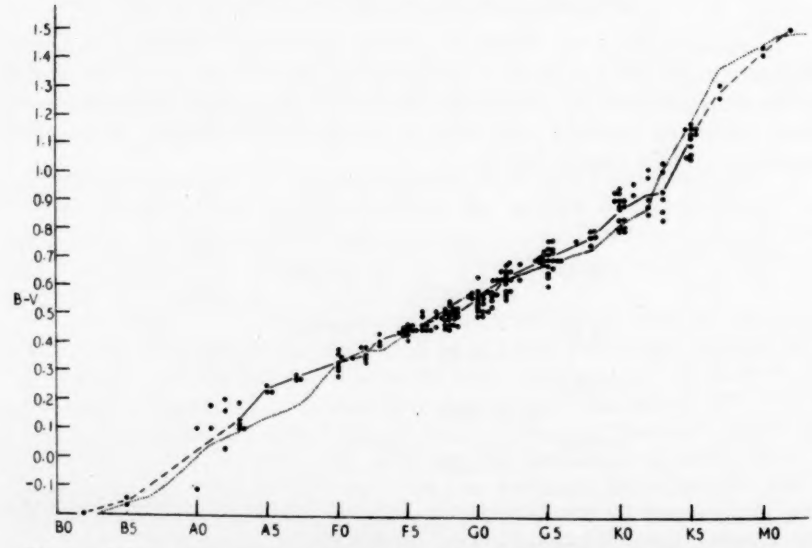

FIG. 1.—*Relation between colour and spectral type for Class V stars. Continuous line : Cape results; dotted line: Morgan and Johnson.*

The general relation between spectral type and colour index is shown in Table III for those spectral classes for which there are at least three stars in Table II. The actual number of stars used for forming the mean colour index is given in the third column and the total range in colour index for these stars in the fourth. The

next three columns show similar data for Johnson and Morgan's work taken from Johnson's paper (2). The correspondence between the two sets of data is far from perfect, but can probably be considered as satisfactory if the limitations of the available data and the variations of colour index within the same nominal type are taken into account.

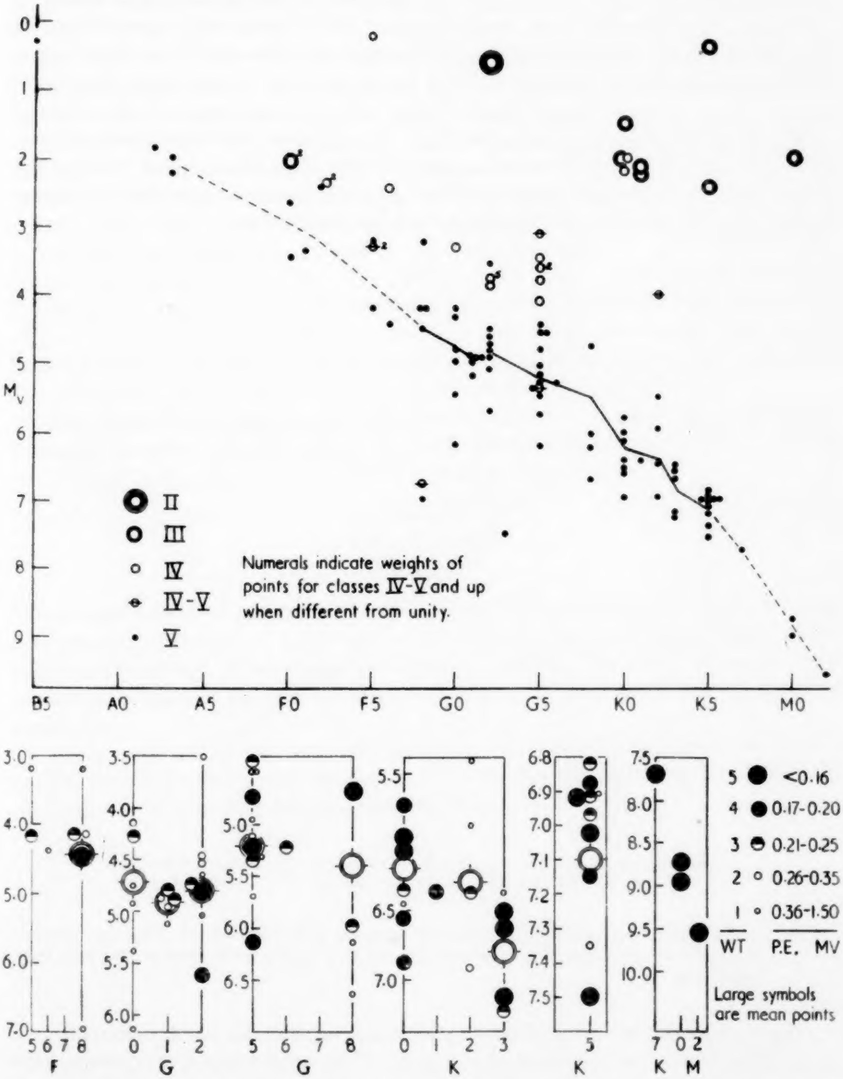

FIG. 2.—*Hertzsprung-Russell diagram for stars of known parallax. The lower portion shows the breakdown of the data for Class V stars in terms of the probable errors of the absolute magnitudes.*

Fig. 1 shows the relation between colour index and spectral type for Class V stars, the only class for which the present data is at all extensive. The dotted line shows the intrinsic colours for class V stars as given by Johnson and Morgan in

Table 14 of their paper (**3**). The continuous line refers to the present Cape results which, for the later type stars, are based on many more stars.

5. *Photometric parallaxes for stars of Class V.*—As a result of their initial selection, many of the stars in Table II have well determined trigonometrical parallaxes. Fig. 2 shows a plot of absolute magnitude against spectral type for the stars for which the probable error of the absolute magnitude is less than $1^{m}\cdot5$ and which are not known to be multiple. The dashed and continuous line indicates the main sequence defined by the stars of Class V. The lower half of the figure shows a breakdown of the data for these Class V stars according to the weight of the absolute magnitudes. Table IV gives the mean of the weighted values of the absolute magnitude for each spectral type for which three or more stars are available, together with comparable data from Keenan and Morgan (**5**), from Gliese (**6**) and from Allen (**7**). The general agreement between the various determinations of absolute magnitude is very satisfactory.

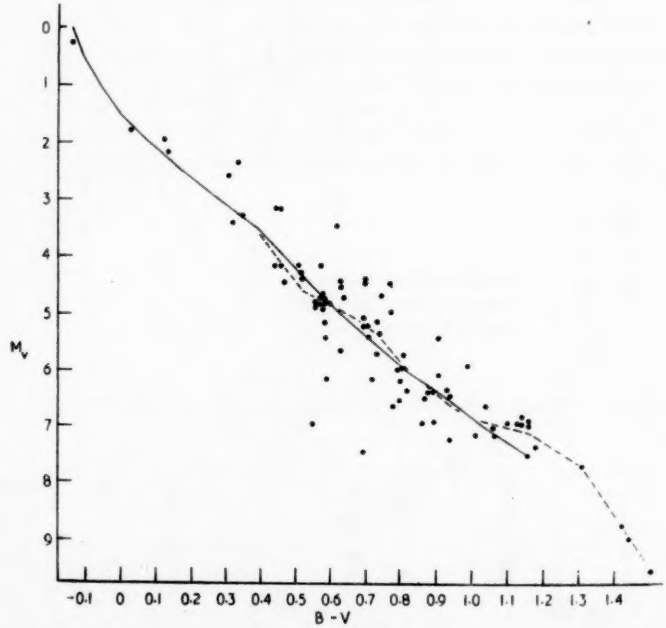

FIG. 3.—*The full line represents the zero-age main sequence of Sandage (Ap. J.,* **125**, *437, 1957); the dotted line the mean relation between* M_V *and B–V used in the derivation of photometric parallaxes.*

Fig. 3 shows a plot of the absolute magnitude against the $B-V$ colours for the same Class V stars as were plotted in Fig. 2. The continuous line represents the main sequence for stars of zero age according to Sandage (**8**). A careful examination of the data indicates that, with very few exceptions, the dispersion in the observed absolute magnitudes for any given colour can be completely attributed to errors in the observed trigonometrical parallaxes. If, then, we assume this dispersion to be purely the result of those errors, we can use the observed mean

curve for deriving the absolute magnitude for any Class V star for which the $B-V$ colour is known.

The result of applying this process to the non-double Class V stars of Table II of types F3 and later is shown in Table V. The mean curve adopted is shown in Fig. 3 by a dotted line and is given implicitly in the second and third columns of Table V. The fourth column of Table V gives the "photometric" parallax derived from the difference between the absolute magnitude given in the third column and the observed magnitude. The observed trigonometrical parallaxes given in the fifth column are taken from the *Yale General Catalogue* (**9**) or from Cape lists (**10**) which were not included in that catalogue.

A comparison of the two sets of parallaxes reveals only five possible stars (R7, 94, 189, 208 and 254) for which there is any reason to suspect a difference that cannot be immediately attributed to the trigonometrical parallax observations, either because these have been observed at one observatory only (usually the Cape) or because the values determined at different observatories differ widely and bracket the value derived from the colour of the star. Stars R7, 94 and 254 appear to be $0^m.9$, $0^m.6$ and $0^m.8$ brighter than would be judged from their colour, while 189 and 208 are $0^m.7$ and $0^m.4$ fainter. Star 189, which is No. 3669 in the Yale Catalogue, has been noted as a subluminous star by Eggen (**11**) while star 208 is part of the system of 36 Oph from which it is separated by 3′.

The space coordinates in parsecs and the space motions in km/s corresponding to the photometric parallaxes are given in the last six columns of Table V. They are relative to the Sun and are referred to axes defined by

$$l = 148^\circ, b = 0^\circ$$
$$l = 58^\circ, b = 0^\circ$$
$$b = 90^\circ$$

The annual proper motions used for computing the space motions are given in the fifth and sixth columns of Table V. In general they are the weighted means of the values given in the *Albany General Catalogue* (**12**) and the values given in more recent catalogues or those derived in the course of the trigonometrical parallax measures.

Stars 2, 28, 32, 68, 81, 200, 227, 278 and 323, which do not appear in Gliese's recent catalogue (**13**), may be closer than 20 parsecs and may be worth the attention of trigonometrical parallax observers.

TABLE I

	HD	GC	M.W.C.			Cape	Previous observations					
			No.	Sp.	Vel.		Bonn	Cape	Lick	Simeis	Vic.	Mt W
R 1	693	190	92	dF5	+14·8*a*	+ 14·1		+13·9	+ 15·0			+13·2
R 3	4128	865	396	gG6	+13·1*a*	+ 13·4	+14·8	+12·9	+ 13·3			
	8779	1738	814	gK0	− 6·4*b*	− 4·4						− 5·9
	22484	4313	1998	dF9	+27·9*a*	+ 27·2	+28·3		+ 28·3	+25·3		+26·2
	35410	6654	3246	gK0	+20·5*a*	+ 19·9		+21·2	+ 20·0			+21·6
	44131	8137	4017	gM1	+47·0*a*	+ 47·9		+47·5	+ 46·6			+48·3
	51250	9103	4551	M0	+20·0*a*	+ 17·8		+20·8	+ 19·3			
	66141	10891	5344	gK3	+70·9*a*	+ 70·7	+72·3		+ 71·2		+68·4	+72·1
R 5	80170	12821	6061	K5	0·0*a*	− 0·3		− 0·2	+ 0·4			
	92588	14694	6655	sgK1	+43·0*a*	+ 42·2						+43·5
	107328	16828	7398	gK1	+35·3*a*	+ 36·2			+ 35·3		+35·2	+35·5
	115521	17995	7902	gM2	−26·9*a*	− 26·6			− 26·7		−27·0	−28·2
	119971	18627	8134	K5	+30·4*a*	+ 26·1		+30·8	+ 30·3			
	126053	19397	8401	dG3	−17·5*b*	− 19·5					−19·2	−17·7
	136202	20591	8861	dF6	+53·5*a*	+ 53·4	+53·4		+ 53·9		+51·4	+50·6
	154417	23050	9855	dF8	−17·5*a*	− 17·6				−17·0	−18·0	−18·6
R 8	157457	23552	10057	K1	+17·6*a*	+ 16·9		+17·3	+ 18·0			
R 9	171391	25374	11031	gG7	+ 6·6*a*	+ 7·5		+ 6·7	+ 6·7			
	173009	25610	11158	gG5	−10·6*a*	− 14·1		−11·6	− 10·0			
	182572	26809	11861	dG7	−99·8*a*	−101·3			−100·6		−99·8	−98·2
	187691	27480	12200	dF8	− 0·1*a*	− 0·3			+ 0·7		− 2·9	− 0·4
R10	203638	29953	13459	gK2	+22·0*a*	+ 22·0		+22·3	+ 21·9			+23·0
	212943	31377	14113	sgK0	+53·8*a*	+ 54·4			+ 54·0		+53·8	+53·9
	223311	33039	14902	gK4	−20·5*b*	− 20·5						−20·0
R11	223647	33107	14930	G7	+14·5*b*	+ 13·4			+ 14·5			

Notes

HD 51250. μ CMaj. A 5605. 4·7 : 8·0, 3″, 340°. Unsuitable as a standard as the spectrum will be composite in small instruments.
HD 119971. Variable velocity suspected.
HD 182572. Velocity probably variable.
HD 223311. This star was included in the 1935 IAU list of standards, but was dropped from later lists on account of suspected variability.

TABLE II

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
1	105	00 03·3	42 02	7·52	+0·58	G0 V	+ 3·2 ± 1·0	7
2	834	10·1	27 08	7·98	+0·95	K0 V	+ 6 1	4
3	1187	13·6	31 43	5·68	+1·36	K5 III	+ 26·2 0·4	6
4	2151	23·1	77 32	2·81	+0·61	G2 IV	+ 23·3 0·2	6
5	2885	29·3	63 14			A2 V	+ 8·2 0·5	5
6	3443	34·8	25 03	5·58	+0·70	G5 V	+ 16·0 ± 0·8	4
7	3737	37·4	41 56	8·48	+0·54	F8 V	+ 17·3 0·8	7
8	4229	39·6	85 58	6·81	+1·28	K5 III	+ 12·0 0·5	7
9	4309	42·3	74 32	7·58	+0·47	F8 V	+ 25 1	4
10	4293	42·6	42 57	5·94	+0·28	A7 V	+ 1 2	5
11	5133	50·6	30 38	7·17	+0·94	K3 V	+ 11·0 ± 0·9	6
12	6269	01 00·9	29 48	6·29	+0·94	G5 IV	+ 56·5 0·6	4
13	6334	01·3	60 22	6·82	+0·46	F5 V	+ 12·2 1·0	5
14		01·3	60 22			F5 V	+ 13·0 1·0	5
15	6623	03·5	72 00	7·36	+1·10	K3 III	+ 6·6 0·5	6
16	6996	07·3	56 52	7·14	+0·44	F5 IV	+ 12·6 ± 0·6	7
17	8224	18·5	57 36	7·01	+0·52	F7 V	− 20·1 0·5	4
18	8391	20·2	43 52	7·01	+0·30	F0 IV	+ 4·2 1·4	5
19	8638	22·4	28 05	8·30	+0·69	G3 V	+ 87·2 1·1	4
20	9379	28·9	59 55	7·96	+0·57	G0 V	− 1·2 1·0	5
21	9468	29·7	59 51	7·97	+0·45	F5 V	+ 4·1 ± 1·1	6
22	9770	32·7	30 10	7·14:	+0·92	K3 V	+ 34·2 1·1	7
23	10360	37·9	56 27	5·07	+0·88	K0 V	+ 20·1 0·4	5
24	10361					K0 V	+ 22·0 0·4	4
25	11695	51·7	46 33	4·4	+1·58	M4 III	+ 5·0 0·4	4
26	12438	59·0	30 14	5·35	+0·88	G5 III	+ 26·4 ± 0·3	5
27	13445	02 08·4	51 04	6·12	+0·81	K0 V	+ 55·3 0·6	5
28	13435	08·5	28 27	7·06	+1·02	K2 V	+ 39·5 0·8	5
29	14412	16·7	26 11	6·34	+0·72	G5 V	+ 7·9 0·6	5
30	15339	24·9	46 13	7·14	+1·11	K0 III	− 1·6 0·5	4
31	15590	27·2	42 18	7·97	+0·66	G5 IV	+ 36·8 ± 0·6	7
32	16591	36·3	42 07	7·25	+0·94	K0 V	+ 48·5 0·8	8
33	17155	41·8	46 39	9·04	+1·05	K5 V	+ 34·7 1·0	4
34	17865	48·9	44 17	8·15	+0·55	F8 V	+ 30·5 1·1	6
35	17926	49·8	31 01	6·39	+0·47	F8 IV–V	+ 8·1 0·4	4
36	18819	58·6	27 50	7·60	+0·58	G0 V	− 1·6 ± 0·8	5
37	18907	59·5	28 17	5·88	+0·79	G5 IV	+ 39·9 0·4	4
38	19743	03 06·6	61 54	7·06	+0·93	G5 IV	− 6·5 0·6	4
39	20052	09·2	62 33	8·32	+0·57	G0 V	+ 26·3 1·3	5
40	20010	09·9	29 11	3·85	+0·51	F8 IV	− 19·4 0·3	4
41	20766	16·7	62 46	5·53	+0·63	G2 V	+ 12·3 ± 0·5	5
42	20807	17·1	62 42	5·26	+0·58	G1 V	+ 11·3 0·4	4
43	20855	17·7	58 10	7·43	+1·05	K0 IV	− 9·5 0·5	5
44	21749	26·1	63 40	8·08:	+1·16:	K5 V	+ 57·3 0·9	4
45	21899	28·5	41 32	6·11	+0·47	F8 V	+ 16·2 0·4	4

Table II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
46	22359	03 31.7	60 45	7.55	+0.48	F8 IV–V	+ 0.7 ± 1.2	5
47	22262	31.9	31 15	6.19	+0.47	F5 V	− 32.6 0.5	5
48	23484	42.3	38 27	6.98	+0.89	K0 V	+ 31.9 1.4	4
49	23817	43.6	64 58	3.82	+1.16	K0 IV	+ 48.6 0.6	4
50	24450	47.4	74 40	8.00	+0.37	F0 V	+ 5.4 0.8	5
51	24512	48.0	74 24	3.31	+1.61	M0 III	+ 14.7 ± 0.4	4
52	24331	48.9	42 43	8.59	+0.91	K2 V	+ 22.4 0.5	4
53	25587	04 00.7	27 37	7.39	+0.52	F8 V	+ 7.8 0.5	5
54	25945	03.6	27 47	5.57	+0.32	F0 V	+ 66.4 0.6	5
55	27274	14.7	53 26	7.62	+1.14	K5 V	− 23.2 0.8	5
56	28454	25.6	47 03	6.10	+0.45	F8 V	+ 17.6 ± 1.1	4
57	29875	39.0	41 58	4.45	+0.33	F2 V	+ 1.7 0.4	6
58	30003	39.5	59 02	6.53	+0.68	G5 V	+ 10.1 0.5	4
59	30361	43.3	47 30	8.33	+0.59	G1 V	+ 59.1 0.8	4
60	31081	45.8	76 24	7.73	+0.48	F8 IV–V	− 0.9 0.4	5
61	30694	46.1	44 21	8.12	+0.32	F2 IV	+ 9.2 ± 1.3	4
62	31093	49.7	34 59	5.85	+0.10	A1 Vn	+ 23 2	4
63	31975	53.9	72 29	6.26	+0.50	F8 V	+ 24.9 0.4	5
64	33563	05 03.9	76 42	7.52	+0.47	F5 V	+ 18.8 0.8	4
65	33262	04.7	57 32	4.69	+0.51	F8 V	− 2.0 0.4	4
66	34554	15.1	31 20	7.46	+0.45	F6 V	+ 11.7 ± 1.3	4
67	36137	25.9	46 08	7.70	+0.40	F3 V	+ 0.8 1.3	4
68	36435	27.1	60 27	6.95	+0.77	G5 V	+ 12.9 0.4	5
69	36553	28.8	47 07	5.46	+0.62	G3 IV	+ 16.5 0.4	5
70	37655	36.5	43 00	7.43	+0.59	G0 V	+ 33.5 1.2	7
71	37706	36.8	46 08	7.32	+0.77	G5 V	+ 11.4 ± 0.5	4
72	38940	45.5	45 40	7.40	+0.49	F6 IV–V	+ 35.5 0.8	5
73	39280	47.9	44 42	7.74	+0.94	G8 IV	+ 29.5 0.7	4
74	39655	50.3	44 02	8.55	+0.35	F2 V	+ 29.3 0.8	5
75	39962	52.3	42 14	7.96	+0.39	F2 V	+ 8.2 0.8	4
76	40292	53.7	52 39	5.28	+0.31	F0 V	+ 28.4 ± 0.7	4
77	41172	06 00.1	27 25	7.11	+0.44	F5 IV–V	− 5.4 0.8	5
78	41534	02.5	32 10	5.64	−0.20	B2 V	+ 105.1 1.8	5
79	43834	11.7	74 44	5.07	+0.71	G5 V	+ 34.8 0.7	4
80	48676	40.9	42 31	7.94	+0.59	G0 IV–V	+ 19.5 0.9	7
81	50223	48.5	46 34	5.13	+0.45	F5 V	+ 20.9 ± 0.8	4
82	50503	49.7	47 16	7.32	+1.16	K2 III	+ 31.3 0.8	4
83	50806	51.6	28 28	6.02	+0.72	G5 IV	+ 70.9 0.6	4
84	51608	53.9	55 12	8.17	+0.77	G7 V	+ 40.1 0.7	5
85	52395	57.8	29 38	7.79	+0.53	G0 V	+ 18.0 0.9	4
86	52698	59.2	25 53	6.72	+0.91	K0 V	+ 12.6 ± 0.6	4
87	53349	07 00.3	58 52	6.02	+0.29	F0 V	+ 10.0 0.7	4
88	56737	13.7	60 59	7.16	+0.41	F3 V	+ 8.7 0.6	4
89	57095	16.1	46 54	6.69	+0.99	K2 V	+ 59.0 0.8	3
90	58895	23.3	58 24	6.59	+0.70	G5 IV	+ 19.7 0.5	4

TABLE II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
91		07 27·7	43 12			G5 V	+ 87·9 ± 0·8	5
92	62644	41·4	45 03	5·06	+0·77	G5 IV	+ 32·4 1·0	5
93	62850	41·7	59 11	7·18	+0·63	G2 IV	+ 15·5 0·7	4
94	63077	43·7	34 04	5·36	+0·57	G0 V	+108·0 0·4	4
95	63685	45·7	61 19	7·42	+0·74	G5 V	+ 27·9 0·7	5
96	63744	46·9	46 57	4·71	+1·07	K0 III	− 1·6 ± 0·3	5
97	64379	50·4	34 35	5·03	+0·44	F5 V	+ 27·2 0·8	4
98	68978	08 11·6	31 35	6·68	+0·61	G5 IV–V	+ 48·5 0·8	4
99	71243	19·9	76 46	4·06	+0·38	F6 IV	− 13·6 0·9	4
100	74543	39·0	73 54	6·82	+1·04	K0 IV	+ 36·1 0·6	4
101	74558	41·1	46 38	6·94	+0·25	A7 III	+ 18 ± 2	4
102	74576	41·4	38 42	6·55	+0·93	K1 V	+ 11·3 0·8	6
103	74842	43·0	42 27	7·20	+0·74	G5 V	+ 5·0 1·1	6
104	76483	53·4	27 29	4·87	+0·13	A3 V	+ 6·2 0·7	4
105	77140	57·2	47 02	5·17	+0·25	F0 III	+ 16·2 0·8	7
106	81830	09 24·2	61 44	5·76	+0·16	A2 Vn	+ 12 ± 2	4
107	81783	24·3	47 33	7·71	+1·28	K3 III	+ 36·0 0·7	4
108	82241	27·6	44 19	6·96	+0·48	F8 V	+ 22·1 1·1	4
109	82434	28·7	40 15	3·57	+0·35	F2 IV	+ 4·2 0·8	5
110	84117	40·0	23 41	4·95	+0·52	G0 V	+ 33·5 0·3	4
111	84850	44·4	58 34	6·22	+0·44	F6 IV–V	+ 5·9 ± 0·8	4
112	85512	49·1	43 15	7·65	+1·18	K5 V	− 9·6 0·8	5
113	86006	52·6	45 30	8·16	+0·71	G5 IV	+ 20·4 1·2	4
114	87638	10 03·4	33 09	7·00	+0·30	F3 IV	+ 1·8 0·9	5
115	87783	04·2	47 08	5·07	+0·88	K0 IV	+ 22·8 0·3	5
116	88201	07·3	32 36	7·46	+0·54	G0 IV	+ 8·5 ± 1·1	6
117	88746	11·0	47 14	8·13	+0·78	G8 V	+ 0·3 1·1	4
118	88742	11·2	32 47	6·38	+0·58	G1 V	+ 42·2 0·6	6
119	90518	24·0	42 29	6·13	+1·13	K1 III	+ 23·2 0·8	6
120	91881	33·7	26 25	6·28	+0·45	F6 V	− 23·9 0·6	5
121	91981	34·2	47 35	7·31	+0·56	G0 IV	+ 31·0 ± 0·7	7
122	92449	37·3	55 21	4·58	+0·79	G2 II	+ 20·2 0·5	5
123	93497	44·6	49 09	2·69	+0·90	G5 III	+ 4·4 0·6	6
124	94444	51·2	44 09	8·12	+0·49	F8 IV–V	+ 6·4 1·3	4
125	95456	58·3	31 34	6·07	+0·50	G0 V	− 2·4 0·7	5
126	96557	11 04·8	32 19	6·58	+0·34	F2 IV	− 1·9 ± 0·6	4
127	96700	05·5	29 54	6·54	+0·58	G2 V	+ 13·5 0·6	6
128	97840	12·6	33 03	7·00	+0·34	F5 IV	+ 0·4 0·7	6
129	98718	18·7	54 13	3·89	−0·17	B5 Vn	+ 13 3	5
130	98818	19·4	60 57	7·35	+1·13	K0 IV	+ 36·9 0·8	4
131	99279	22·5	61 22	7·22	+1·27	K7 V	+ 4·7 ± 0·8	5
132	100623	32·1	32 34	5·97	+0·80	K0 V	− 21·8 0·4	4
133	100733	32·8	47 06	5·75:	+1·66:	M3 III	+ 18·4 0·6	6
134	100773	32·9	60 37	6·57	+0·35	F2 IV	− 0·8 1·7	5
135	101349	37·0	48 12	8·99	+0·84	K0 V	− 2·9 0·7	4

Table II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)		Pl.
		h m	° ′						
136	101408	11 37·4	45 32	7·25	+1·04	G8 IV	+ 28·1	± 0·7	5
137	101581	38·6	44 08	7·77	+1·07	K5 V	+ 13·3	0·8	4
138	102350	44·1	60 54	4·10	+0·90	G3 III	− 3·0	0·4	4
139	102365	44·1	40 14	4·91	+0·64	G5 V	+ 14·7	0·6	4
140	103746	54·2	46 48	6·27	+0·39	F3 IV–V	+ 7·6	1·6	5
141	103932	55·5	27 25	6·97	+1·16	K5 V	+ 46·7	± 0·9	4
142	103975	55·7	47 42	6·77	+0·50	G0 V	− 2·1	0·6	4
143	109200	12 30·7	68 29	7·12	+0·82	K0 V	+ 0·3	0·9	5
144	110287	38·6	45 52	5·84	+1·52	K3 II	+ 6·9	0·4	6
145	110304	38·8	48 41	2·17	−0·01	A0 III	− 1·9	0·9	4
146	110477	40·2	60 52	7·77	+0·42	F6 IV	+ 1·9	± 1·1	6
147	110829	42·7	60 42	4·71	+1·06	K1 III	+ 5·2	0·6	5
148	112164	52·2	43 53	5·89	+0·62	G2 IV	+ 32·0	0·5	4
149	115577	13 15·6	28 04	6·81	+0·98	G8 IV	+152·2	0·5	4
150	116243	20·6	64 16	4·53	+0·84	G5 III–IV	+ 12·9	0·4	5
151	117440	28·1	39 09	3·90	+1·20	G8 III	− 1·3	± 0·3	5
152	117558	28·8	27 51	6·47	+0·10	A0 III–V	− 4	2	5
153	119070	39·0	48 13	9·18	+0·77	G5 V	+ 5·0	0·9	4
154	119629	42·4	48 33	6·77	+0·51	F8 V	− 21·7	1·0	4
155	120592	48·4	48 03	7·4	+0·7	G5 V	+ 6·9	1·0	4
156	120593	48·5	48 03	7·5	+0·5	F6 V	+ 4·5	± 1·4	4
157	121416	53·2	46 21	5·82	+1·15	K0 IV	− 1·0	0·6	7
158	121852	56·1	45 14	7·38	+0·50	F7 V	+ 3·8	1·1	5
159	123651	14 06·9	46 02	8·21	+0·53	G1 V	− 22·0	1·1	6
160	123797	07·8	48 33	6·63	+0·91	G5 IV	− 63·2	0·4	4
161	125072	15·5	59 08	6·66	+1·04	K3 V	− 16·1	± 1·1	4
162	125932	20·2	27 32	4·80	+1·34	K5 III	+ 20·9	0·5	6
163	125968	20·5	27 36	7·75	+0·65	G5 IV–V	+ 25·9	0·7	5
164	128266	34·1	45 55	5·36	+1·00	K0 III	− 15·9	0·3	6
165	128582	35·9	46 22	6·08	+0·48	F8 IV–V	− 12·9	0·5	4
166	128931	37·5	29 05	7·82	+0·44	F3 V	+ 27·2	± 0·9	4
167	128898	38·4	64 46	3·18	+0·26	F0 III	+ 6·5	0·4	5
168	129178	39·0	29 09	8·13	+0·47	F6 V	+ 33·0	1·5	4
169	129422	41·3	62 40	5·36	+0·29	A7 Vn	− 5	3	6
170	129747	42·5	45 40	8·49	+0·66	G2 V	− 53·6	1·2	5
171	131342	51·7	59 55	5·22	+1·16	K1 III	− 13·8	± 0·4	6
172	132301	56·5	43 37	6·57	+0·46	F5 V	− 21·9	0·8	4
173	134060	15 06·7	61 14	6·28	+0·61	G3 IV	+ 38·3	0·8	4
174	134331	07·4	43 32	7·01	+0·61	G5 V	− 3·8	0·7	5
175	135379	13·6	58 37	4·07	+0·10	A3 V	+ 9·1	0·8	5
176	136352	18·4	48 08	5·67	+0·63	G2 V	− 70·9	± 0·8	4
177	138549	30·8	30 51	7·96	+0·70	G5 V	+ 14·4	0·5	4
178	139127	34·7	42 24	4·36	+1·44	M0 III	− 6·4	0·4	6
179	139664	37·8	44 30	4·64	+0·39	F5 IV–V	− 4·8	1·0	6
180	139961	39·4	44 47	8·86	+0·10	A0 V	+145	3	5

TABLE II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
181	140690	15 43·3	43 05	8·10	+0·64	G5 IV	+ 60·5 ± 1·5	4
182	140901	44·2	37 46	6·03	+0·70	G6 V	− 5·9 0·8	5
183	141891	50·7	63 17	2·86	+0·28	F2 IV	+ 2·6 0·7	6
184	142254	51·8	42 28	6·69	+0·38	F0 V	− 17·0 0·9	4
185	142529	53·5	48 01	6·30	+0·36	F2 V	+ 1·8 0·7	5
186	143474	59·5	57 38	4·64	+0·23	A5 V	− 8 ± 2	4
187	144628	16 05·7	56 19	7·12	+0·84	K3 V	+ 37·9 1·0	6
188	145158	07·8	45 12	6·65	+0·47	F8 V	− 5·0 0·9	4
189	145417	09·8	57 25	7·54	+0·80	K0 V	+ 11·3 1·2	5
190	145544	10·8	63 34	3·85	+1·12	G2 II	− 5·0 0·4	5
191	146624	15·2	28 29	4·79	+0·03	A2 V	− 13·2 ± 0·9	4
192	146667	15·8	42 33	5·44	+0·10	A3 Vn	− 12 3	6
193	146775	16·0	28 10	7·68	+0·59	G0 V	− 31·1 0·6	5
194	147722	21·5	29 35	5·40	+0·58	G0 IV	− 0·2 0·5	5
195	147723	21·5	29 35			G0 IV	− 2·9 0·6	5
196	148587	28·5	63 44	7·36	+0·57	G0 V	+ 40·6 ± 1·1	4
197	149324	35·9	77 25	4·26	+1·07	K0 III	− 30·9 0·5	3
198	150248	38·2	45 16	7·05	+0·63	G3 V	+ 65·4 0·9	4
199	150437	39·0	29 02	7·86	+0·68	G2 V	+ 12·5 0·8	4
200	151337	45·3	47 38	7·38	+0·91	K0 V	+ 29·4 0·8	4
201	152334	51·1	42 17	3·67	+1·38	K5 III	− 17·9 ± 0·5	5
202	153072	55·5	37 33	6·05	+0·19	A3 V	− 27·3 1·1	7
203	152923	55·6	59 15	7·12	+0·45	F6 V	− 13·4 1·0	4
204	153075	56·3	57 13	7·01	+0·56	G0 V	+ 98·7 0·9	5
205	153950	17 00·9	43 14	7·41	+0·55	G2 IV–V	+ 30·7 0·9	6
206	154577	05·7	60 40	7·40	+0·89	K0 V	+ 8·8 ± 1·0	4
207	155886	12·3	26 32	4·38	+0·84	K0 V	− 0·6 0·4	5
208	156026	13·2	26 29	6·32	+1·16	K5 V	+ 1·8 0·7	7
209	156274	15·3	46 35	5·48	+0·80	G8 V	+ 24·5 0·4	5
210	156751	18·5	58 25	6·77	+0·25	A5 V	+ 6 1	4
211	157750	23·3	32 56	8·02	+0·66	G2 V	− 18·6 ± 1·2	5
212	157919	24·2	29 49	4·30	+0·39	F5 IV	+ 36·9 0·4	4
213	158311	28·1	62 12	7·50	+0·90	G8 IV	+ 24·6 0·6	4
214	159809	35·2	45 44	7·44	+1·02	K1 IV	+ 6·8 1·0	4
215	159868	35·4	43 07	7·23	+0·71	G5 V	− 23·8 0·6	6
216	160691	40·2	51 49	5·10	+0·70	G5 V	− 9·7 ± 0·5	4
217	162396	49·3	41 59	6·17	+0·52	F8 V	− 14·4 0·8	4
218	165189	18 03·2	43 26	4·92	+0·23	A5 V	− 6 2	8
219	165271	03·7	46 54	7·65	+0·65	G5 IV	− 88·1 0·8	4
220	165696	05·8	45 57	7·33	+0·49	F8 V	− 29·3 0·8	4
221	165753	06·1	44 57	7·02	+1·11	K0 IV	− 0·7 ± 0·8	5
222	166348	08·7	43 27	8·37	+1·31	K7 V	− 2·0 0·6	4
223	167576	13·8	27 44	6·66	+1·27	K3 III	− 10·4 1·0	6
224	167665	14·2	28 18	6·35	+0·52	G0 V	+ 6·9 0·5	8
225	169233	21·8	30 47	5·58	+1·16	K0 III–IV	− 18·7 0·4	6

TABLE II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	*V*	*B–V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
226	164461	18 25·8	87 39	5·29	+1·30	K3 II	+33·3±0·6	5
227	171627	33·9	28 33	6·77	+0·97	K1 V	+24·7 0·9	4
228	173339	43·2	43 51	7·39	+0·27	A7 V	−17 5	4
229	174153	47·7	44 32	7·54	+0·52	G0 V	+67·4 0·9	4
230	175329	54·2	60 16	5·15	+1·37	K1 III–IV	+179·6 0·8	5
231	175856	55·8	44 12	8·57	+0·52	F6 V	+3·3±0·7	5
232	176687	59·4	29 57	2·59	+0·10	A2 III	+28·2 0·8	8
233	176578	59·5	47 07	6·85	+0·97	K0 IV	−7·1 0·8	4
234	177474	19 03·0	37 08	4·20	+0·50	F8 V	−50·8 0·3	4
235						F8 V	−53·2 0·3	4
236	179140	10·5	58 05	7·21	+0·63	G2 V	+30·3±1·0	4
237	181428	18·5	29 42	7·09	+0·56	G0 IV	+36·4 0·9	4
238	181544	18·9	29 37	7·08	+0·57	G0 IV	+36·4 0·6	5
239	181773	21·2	62 17	7·59	+0·46	F5 IV	−32·9 1·1	8
240	182156	21·4	30 54	7·63	+1·17	K0 IV	+14·8 0·9	4
241	183216	26·5	30 54	7·12	+0·59	G2 V	−42·8±0·6	5
242	183312	27·0	32 12	6·56	+0·39	F5 IV–V	+8·3 0·7	4
243	183877	29·6	28 07	7·14	+0·68	G5 IV	−40·0 0·8	6
244	186651	43·9	43 28	7·11	+0·55	G0 V	+33·5 1·0	5
245	186975	45·9	45 52	7·25	+1·10	K0 IV	−24·4 0·4	4
246	188555	54·2	45 58	8·60	+0·39	F5 IV	−41·3±1·2	5
247	188228	54·9	73 03	3·93	−0·01	A0 V	−4 1·7	6
248	188815	55·4	46 14	7·46	+0·45	F6 V	−6·4 0·8	5
249	189198	57·3	45 15	5·80	+0·28	A7 III	−6 1·5	4
250	189247	57·4	44 07	7·66	+0·42	F5 IV	−22·5 1·0	5
251	189567	20 00·6	67 27	6·08	+0·63	G2 V	−11·7±0·6	4
252	190333	02·8	43 21	9·20	+0·66	G2 V	+58·2 1·0	5
253	189899	03·2	74 22	7·61	+0·50	F8 IV–V	−2·9 1·2	4
254	190248	03·8	66 19	3·54	+0·75	G8 V	−21·2 0·2	4
255	190779	05·1	46 27	8·23:	+0·42:	F5 V	−43·0 0·8	5
256	191408	07·9	36 14	5·30	+0·87	K3 V	−128·9±0·3	6
257	191584	09·0	42 56	6·21	+1·25	K2 III	−0·8 0·5	5
258	191849	10·2	45 19	7·96	+1·44	M0 V	−20·8 0·8	5
259	192310	12·2	27 11	5·72	+0·91	K0 V	−54·2 0·3	5
260	194433	23·7	37 34	6·23	+0·98	K2 IV–V	+25·2 0·4	4
261	194640	24·6	31 02	6·59	+0·73	G5 V	0·0±0·7	6
262	196081	32·9	26 57	7·19	+0·41	F5 IV	−26·3 0·8	7
263	196067	35·8	75 32			G1 V	−12·5 0·8	4
264	196068	35·9	75 31	6·02	+0·62	G5 V	−11·2 1·0	5
265	196051	35·9	76 22	6·00	+0·45	F5 IVn	−36 2	7
266	196531	36·0	28 36	7·94	+0·53	F8 V	−32·6±0·8	4
267	196917	38·3	31 47	5·76	+1·55	M0 III	−97·3 0·9	7
268	197214	40·2	29 36	6·95	+0·67	G5 V	−19·8 0·7	4
269	197484	42·2	43 08	8·86	+0·63	G2 V	+21·1 0·9	4
270	197416	42·6	60 28	8·15	+0·48	F8 V	−3·6 0·8	4

TABLE II (*cont.*)

No.	HD	(1950) R.A.	(1950) S. Dec	*V*	*B*−*V*	Spec.	Vel. (km/s)	Pl.
		h m	° ′					
271	197900	20 44·9	44 23	6·46	+1·18	K1 IV	− 14·8 ± 0·5	6
272	197937	45·1	44 10	5·10	+0·35	F1 V	− 11 2	6
273	198828	51·4	46 47	7·39	+0·53	F8 V	− 9·5 1·1	4
274	199288	54·4	44 19	6·50	+0·58	G0 V	− 8·2 0·4	9
275	200011	58·9	43 12	6·63	+0·68	G3 IV	− 30·5 0·8	4
276	200026	59·0	43 12	6·89	+0·97	K0 IV	− 32·6 ± 0·7	5
277	202560	21 14·3	39 04	6·68	+1·42	M0 V	+ 19·4 0·7	6
278	202457	14·8	61 33	6·58	+0·70	G5 V	− 22·1 0·7	4
279	203448	20·1	31 02	7·86	+0·56	G0 IV	− 35·8 1·0	4
280	203608	22·3	65 36	4·23	+0·47	F8 V	− 28·8 0·4	5
281	203985	23·8	45 02	7·45:	+0·94:	K0 V	+ 8·2 ± 0·6	6
282	205067	30·6	28 07	7·61	+0·65	G2 V	− 26·6 0·8	5
283	205153	31·2	28 07	8·23	+0·54	G0 IV	− 15·0 0·9	7
284	205390	33·3	51 04	7·17	+0·89	K2 V	+ 28·1 0·6	5
285	206341	39·2	27 55	7·68	+1·08	K0 IV	− 21·3 0·9	4
286	206395	39·8	43 43	6·69	+0·53	G0 IV	− 21·9 ± 0·6	5
287	206868	42·7	28 57	7·69	+0·39	F2 V	− 17 2	5
288	206948	43·7	46 37	7·56	+1·18	K3 III	− 33·1 0·6	4
289	207155	44·8	31 08	5·03	+0·06	A2 IIIn	+ 12 2	5
290	207129	45·0	47 32	5·60	+0·59	G2 V	− 6·5 0·6	5
291	207964	51·5	62 07	5·91	+0·40	F0 IV	+ 1 ± 2	8
292	208215	52·8	47 10	6·55	+0·45	F5 V	+ 24·8 0·5	4
293	208812	56·8	43 43	8·17	+0·47	F8 IV–V	− 6·3 1·2	7
294	209100	59·6	57 00	4·73	+1·06	K5 V	− 40·4 0·3	7
295	209742	22 03·6	45 38	8·46	+0·86	K2 V	− 23·4 0·8	4
296	209952	05·1	47 12	1·73:	−0·14	B5 V	+ 9 ± 3	4
297	210193	06·6	41 28	7·87	+0·65	G5 V	+ 8·0 0·8	6
298	210441	08·4	44 05	6·63	+1·00	G8 IV	− 13·3 0·4	4
299	211415	15·0	53 52	5·40	+0·59	G1 V	− 13·2 0·4	5
300	211998	20·4	72 30	5·29	+0·64	G0 V	+ 29·9 0·5	5
301	213042	26·4	30 16	7·68	+1·10	K5 V	+ 8·6 ± 0·7	5
302	213135	27·0	27 22	5·99	+0·33	F0 V	+ 3·2 1·0	4
303	214094	33·8	43 44	6·77	+0·52	F6 V	− 48·6 0·5	5
304	214308	35·1	46 58	7·70	+0·41	F5 IV	− 3·8 1·0	7
305	214385	35·4	27 42	7·92	+0·62	G2 V	+ 46·3 1·1	5
306	214690	37·6	30 55	5·89:	+1·31	K3 III	+ 81·0 ± 0·6	5
307	214749	37·9	29 56	7·86	+1·14	K5 V	+ 1·0 1·2	6
308	214759	38·1	32 15	7·41	+0·80	G8 V	− 21·0 1·0	5
309	214953	39·7	47 28	5·99	+0·56	G1 V	+ 17·3 0·6	4
310	216042	47·2	33 04	6·36	+0·30	F2 IV	+ 24·0 0·6	6
311	216054	47·3	41 45	7·80	+0·73	G5 V	+ 21·6 ± 0·7	4
312	216743	53·1	42 49	7·28	+0·18	A1 Vn	+ 7 3	4
313	216803	53·6	31 50	6·47	+1·13	K5 V	+ 15·1 0·9	5
314	216956	54·9	29 53	1·18	+0·12	A3 V	+ 5·3 1·0	5
315	216989	55·2	45 26	7·73	+0·36	F0 V	− 5 1	5

TABLE II (*cont.*)

No.	HD	(1950) R.A.	S. Dec	V	$B-V$	Spec.	Vel. (mk/s)	Pl.
		h m	° ′					
316	217595	22 59·6	45 34	7·22	+0·43	F5 V	+ 20·5 ± 0·3	7
317	217766	23 00·7	43 21	7·78	+0·53	F8 V	+ 34·0 1·1	5
318	217816	01·1	46 26	8·13	+0·50	F6 IV–V	+ 5·0 0·8	5
319	217987	02·7	36 09	7·36	+1·50	M2 V	+ 8·7 0·8	5
320	218227	04·1	43 47	4·29	+0·44	F6 IV	+ 4·7 1·0	4
321	219249	12·2	57 00	7·97	+0·69	G5 V	− 5·8 ± 0·7	4
322	219409	13·1	30 07	6·53	+1·09	K1 III	− 47·7 0·6	5
323	219482	14·0	62 16	5·67	+0·49	F8 V	0·0 0·3	4
324	219709	15·7	58 35	7·33	+0·69	G2 V	+ 20·8 0·8	7
325	220881	24·9	27 33	7·46	+0·28	F0 III	− 18·7 1·3	5
326	221818	32·9	47 13	8·57	+0·78	G8 V	− 5·0 ± 1·0	5
327	221839	33·0	27 46	6·68	+0·55	G0 IV	+ 1·8 0·3	5
328	222237	36·7	72 59	7·09	+1·01	K3 V	+ 70·1 0·8	6
329	222412	37·8	26 29	7·60	+0·43	F6 IV	− 4 1	7
330	222433	38·0	32 21	5·33	+0·97	K0 III	+ 15·6 0·3	5
331	222480	38·5	32 21	7·13	+0·66	G5 IV	+ 21·5 ± 0·9	5
332	222508	38·8	41 51	7·82	+0·46	F7 V	+ 10 2	5
333	222803	41·4	45 22	6·11	+0·99	G8 IV	− 30·8 0·6	4
334	223065	43·9	41 51	7·0	+0·2	A2 V	− 33 2	9
335	223633	48·8	42 39	7·57	+0·44	F5 IV–V	+ 22 1	9
336	224296	54·1	42 28	7·90	+0·41	F5 IV–V	+ 6·1 ± 1·1	6
337	224360	54·7	46 23	7·72	+0·44	F5 V	− 2 1	6
338	224750	57·8	44 34	6·30	+0·75	G3 IV	+ 3·0 0·5	5
339	224889	59·1	77 20	4·79	+1·29	K2 III	+ 22·4 0·3	6
R2	1581	00 17·5	65 10	4·23	+0·56	G2 V	+ 9·5 ± 0·4	4
R4	20794	03 17·9	43 16	4·24	+0·70	G5 V	+ 87·0 0·3	4
R6	101021	11 34·7	61 00	5·15	+1·13	K1 III	+ 3·3 0·5	4
R7	114837	13 11·1	58 50	4·92	+0·46	F8 V	− 64·8 1·0	5

Notes

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4 β Hyi.

5 β_2 Tuc. Bi. 4·9 : 5·9, 41·3 y; a=0″·48, p_d=0″·024. Mean date of rad. vel. obs. 1954.22. ($\beta_1+\beta_2$ Tuc. Bi. 4·48 : 4·52, 27″·1, 170°).

6 ADS 520 Bi. 6·3 : 6·4, 25·0 y; a=0″·67, p_d=0″·063. Mean date of rad. vel. obs. 1953.98. Single lines only seen in spectra.

10 Difficult spectrum, broad lines.

11 The present plates do not support the designation as a sp. bi. in the Mount Wilson Cat. (No. 500).

13, 14 Bi. 7·6 : 7·6, 5″·0, 128°. No certain motion. C : 11 mag., 100″, 41°. Joint magnitude and colour.

16 Bi. 7·9 : 8·2, 0″·3, 314° (1945). Joint spectrum.
17 Bi. 7·2 : 10, 2″·6, 229° (1942). p_d=0″·019.
18 Difficult spectrum, broad lines.
19 Eggen gives mag. as 7·74. Knox-Shaw from southern Harv. vis. obs. gives 8·21.
22 Triple 7·8 : 7·9 : 11·2; AB a=0″·17, 4·6 y ; AB, C a=1″·58?, 144 y?. See O. J. Eggen (*P.A.S.P.*, **64,** 230, 1952) p_d=0″·051. Mag. prob. var. A fourth star D of mag. 7 is such that AB, D is 140″, 20°.
23, 24 p Eri. Bi. 5·8 : 5·9, 244 y, a=8″·36; p_d=0″·165. See J. L. Dessy (*Pub. La Plata*, **20,** No. 3, 1949); also G. B. van Albada (*Bosscha Cont.* No. 5, 1956). The suggestion that these stars are peculiar does not seem well founded. The secondary and primary (Nos. 23 and 24 respectively) are Nos. 770 and 771 in the list of parallax stars given in *Cape Annals*, XV, 1952. The secondary is 0·16 mag. fainter than the primary, giving, on the assumption of identical colours for the components, visual magnitudes of 5·91 and 5·75. The joint colour is consistent with the spectral types. The deduced photometric parallaxes are 0″·129 and 0″·120 for secondary and primary, while the observed Cape relative parallaxes are 0″·138 and 0″·129.
25 ψ Phe. Mag. var. through 0^m·12. H. Spencer-Jones (*Cape Ann.*, **10,** Pt. 8, 1928) suspected var. vel. Present rad. vel obs. all close to mean date 1952·58.
26 π For.
37 ϵ For.
40 α For. ADS 2402 Bi. 4·0 : 6·5, 154·5 y, a=2″·70, p_d=0″·053. Companion may be var.; Spectra of bright star only. Joint mag. and col.
41 ζ^1 Ret.
42 ζ^2 Ret. Common motion with 41.
44 Mag. and col. measures impeded by neighbouring star of differing p.m.
46 Bi. 8·2 : 8·3, 0″·5, 114° (1955).
47 Bi. 6·7 : 7·3, 19·4 y, a=0″·24, p_d=0″·024. Vel. refers to combined light, mean date 1953·92.
48 Vel. possibly var.
49 β Ret. Mount Wilson Cat. (No. 2114) designates as a sp. bi. Mean date of present obs. 1953·81; var. of vel. not confirmed.
51 γ Hyi.
57 α Cae. Bi. 4·5 : 12, 6″·6, 121° (1933).
58 Bi. 7·1 : 7·3, 2″·1, 91°; p_d=0″·037. Parabolic orbit (*Union Obs. Circ.* 95). Spectra of combined light shows single lines.
62 Interferometric bi., 0″·228, 40°·0 (1956·16). See W. S. Finsen (*Obs.*, **72,** 125, 1952). Lines very broad, only one spectrum seen.
64 Bi. 7·0 : 12, 0″·9, 95°.
65 ζ Dor.
68 H and K lines in emission, +17 km/s.
69 Triple 5·5 : 11·7 : 12·7, A, BC 25″·9, 257°; BC 0″·6, 90°.
70 Vel. probably var.
71 Joint mag and col. with GC 7049. Bi. 7·4 : 9·5, 5″·1, 67° (1941), p_d=0″·037.
79 α Men.
80 Bi. 8·0 : 10·5, 2″·4, 186° (1951).
86 H and K lines in emission, +16 km/s.
89 Bi. 7·1 : 7·9, 0″·16, 115° (1956). H and K lines in emission, +56 km/s.
91 The faint companion of σ Pup I, 22″, common motion. Rad. vel. obs. of σ Pup I (D. S. Evans, *MNASSA*, **16,** 4, 1957) confirm, with only slight modifications, the work of R. E. Wilson (*Lick Obs. Bull.*, **9,** 117, 1918). The stars are too close for the photometry of the faint companion.
92 Vel. possibly var.
96 Q Pup.
97 Bi. 5·1 : 8·1, 3″·3, 275° (1954), p_d=0″·021.
99 α Cha.
101 Difficult spectrum, broad lines.
102 H and K lines in emission, +11 km/s.
104 δ Pyx.

TABLE II (*cont.*)

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97 Bi. 5·1 : 8·1, 3″·3, 275° (1954), p_d=0″·021.

99 α Cha.

101 Difficult spectrum, broad lines.

102 H and K lines in emission, +11 km/s.

104 δ Pyx.

105 Bi. 5·2 : 11, 24″·1, 338° (1929).
106 Bi. 6·0 : 10, 8″·7, 328° (1933). Difficult spectrum, lines broad and faint.
108 This star has a common p.m. with HD 82207 from which it is distant 108″·4 in p.a. 77°.
109 ψ Vel. Bi. 3·9 : 4·8, 34·1 *y*, a=0″·92; p_d=0″·061. Companion reported as variable $4^m·5–5^m·1$. Lines broad. Only one spectrum recorded but its appearance is suspected of variation with lines narrower in 1953. Cape photometric obs. of massed light give no sign of var.
115 Bi. 5·4 : 7·1, 0″·5, 330° (1956).
117 Bi. 8·3 : 10·8, 5·1, 128° (1947).
120 Bi. 6·5 : 7·7, 211 *y*, a=1″·0, p_d=0″·019. Mean date of vel. obs. 1954·70.
121 Bi. 7·6 : 11, 1″·9, 156° (1935).
122 x Vel. A multiple star. AB 4·4 : 6·6, 51″·6, 105° (1938); AD 4·4 : 14, 18″·4, 86° (1880); AE 4·4 : 13, 33″·1, 225° (1880); BC 6·6 : 11, 21″·0, 174° (1934).
123 μ Vel. Bi. 3·0 : 6·5, 0″·65, 90° (1943). This pair has closed in and the companion has been invisible since 1945. Period unknown; p_d=0″·022. Vel. of brighter star only, possibly var.
129 π Cen. Bi. 4·7 : 5·5, 42·8 *y*, a=0″·275, p_d=0″·011.
131 Bi. 7·6 : 8·6, 265 *y*, a=4″·01, p_d=0″·105.
132 Companion, 15^m, 17″, which, according to a note in the Yale " General Catalogue of Parallaxes ", has the same proper motion.
133 Bi. 5·8 : 13, 6″·7, 163° (1949). Possibly an optical double. Mag. and col. probably var.
137 H and K lines in emission, +16 km/s.
138 Bi. 4·1 : 13, 25″, 320° (1929).
140 Difficult spectrum, broad lines.
141 Mount Wilson Cat. (No. 7208) designation as a sp. bi. seems doubtful. Eggen gives mag. as 7·23. Knox-Shaw from southern Harv. vis. obs. gives 6·94.
145 γ Cen. Bi. 3·1 : 3·1, 84·5 *y*, a=0″·930, p_d=0″·024. All obs. refer to combined light. Mean date of vel. obs. 1952·49.
147 ι Cru. Bi. 4·7 : 9·5, 26″·9, 22° (1942).
150 m Cen.
151 d Cen. Bi. 4·6 : 4·8, 62·6 *y*, a=0″·155, p_d=0″·003. Obs. refer to combined light.
152 Difficult spectrum.
155, 156 Bi. 7·4 : 7·5, 30″·5, 131° (1932). A wide pair making measurement of mag. and col. difficult.
162 51 Hya.
164 a Lup. Tri. AB 6·1 : 6·1, 0″·142, 41° (1955·6); AB, C 5·4 : 7·9, 19″·0, 24° (1933). Mean date of vel. obs. is 1954·10. Lines suspected of duplicity in 1953 Feb.
167 α Cir. Bi. 3·2 : 8·6, 16″, 232° (1943).
169 Difficult spectrum, broad lines.
174 Forms a bi. with HD 134330 7·0 : 7·5, 50″·6, 22° (1902). Common p.m.
175 β Cir.
176 ν_2 Lup.
178 ω Lup. Bi. 4·3 : 11, 11″·8, 28° (1933). Present obs. do not support suggestion of var. vel. by H. Spencer Jones (*Cape Ann.*, **10**, Pt. 8, 1928).
179 g Lup. Difficult spectrum, broad lines.
180 Difficult spectrum with diffuse lines. This star is underluminous. See J. Jackson (*MNASSA*, **8**, 29, 1949).
181 Bi. 8·3 : 8·8, 0″·15, 324° (1935).
182 Bi. 6·0 : 13·5, 15″, 133° (1953).
183 β TrA. A 14^m star which s.p. β Tri. by 155″ possibly shares its p.m.
186 ι_1 Nor. Tri. AB 5·6 : 5·7, 26·4 *y*, a=0″·356, p_d=0″·028; AB, C 4·6 : 8, 10″·8, 245°. Difficult spectrum, broad lines. Vel. refers to mean date 1953·12.
190 δ TrA. Bi. 4·0 : 11, 25″, 105° (1892).
191 d Sco.
192 λ Nor. Bi. 6·0 : 6·9, 0″·4, 120° (1956).
194, 195 Bi. 5·94 : 6·59, 5″·7, 353° (1935), p_d=0″·036. Joint mag. and col.

197 β Aps.

201 ζ Sco.

202 Bi. 6·9 : 7·1, 31·2 *y*, $a = 0''{\cdot}169$, $p_d = 0''{\cdot}011$. Spectrum difficult with broad lines lines but no indication of var. vel. or double lines. Mean date of vel. obs. 1953·73.

203 Bi. 7·1 : 11, 26″·1, 12° (1918).

207 36 Oph. Bi. 5·29 : 5·33, 4″·4, 166° (1955). The mag. and col. refer to the joint light but the rad. vel. and spectral type to the north component only. H and K lines in emission, −3 km/s.

208 H and K lines in emission, +2 km/s. Vel. possibly var. This star and 207 have a common motion.

209 41 Ara. Multiple star. AB 5·5 : 8·7, 550 *y*, $a = 8''{\cdot}8$, $p_d = 0''{\cdot}113$. AC 5·5 : 13, 41″·8, 279° (1900); AD 5·5 : 8·5, 47″, 30° (1900). Mag. and col. refer to joint light, rad. vel. and spectrum to bright star only.

210 Bi. 6·8 : 9·6, 9″·4, 283° (1917).

211 Vel. possibly var.

212 45 Oph.

214 Bi. 7·4 : 12, 1″·8, 246° (1948).

216 μ Ara.

218 Bi. 5·8 : 5·8, 214 *y*, $a = 1''{\cdot}208$, $p_d = 0''{\cdot}022$. Difficult spectrum, broad lines. Vel. refers to the combined light, and is possibly, var. Mean date of rad. vel. obs. 1952·74.

222 H and K lines in emission, −3 km/s.

226 χ Oct. Spectra all rather weak.

228 Extremely difficult spectrum, lines very broad.

230 ω Pav.

232 ζ Sgr. Bi. 3·4 : 3·6, 20·8 *y*, $a = 0''{\cdot}520$, $p_d = 0''{\cdot}040$. All data for combined light. Vel. not considered var.

234, 235 γ CrA. Bi. 4·95 : 5·05, 119·3 *y*, $a = 2''{\cdot}07$, $p_d = 0''{\cdot}064$. Vel. of components observed separately, mean date 1953·9.

236 Bi. 7·3 : 9·8, 0″·5, 195° (1942).

239 Bi. 7·7 : 10·8, 2″·9, 53° (1937).

247 ϵ Pav.

249 Difficult spectrum, broad lines.

252 Bi. 9·4 : 11, 1″·7, 299° (1934).

254 δ Pav.

255 With HD 190803 forms Bi. 8·2 : 9·2, 63″·1, 102°. This is probably an optical pair only as the p.m.s differ. Mag. and col. not sufficiently observed to deserve standard weight.

256 Bi. 5·3 : 11·5, 7″·1, 123° (1949).

259 Mount Wilson Cat. (No. 12602) designation as sp. bi. not supported.

260 Bi. 6·5 : 8·2, 0″·25, 245° (1952), $p_d = 0''{\cdot}024$. All data refer to joint light. Mean date of vel. obs. 1953·69.

263, 264 Bi. 7·08 : 7·60, 17″·2, 17° (1940). Joint mag. and col.

265 μ_1 Oct. Difficult spectrum with broad lines.

272 ι Mic. Bi. 5·1 : 15, 4″·3, 271° (1932). Difficult spectrum, broad lines.

273 Bi. 7·9 : 8·4, 0″·4, 138° (1952). All data for combined light.

275, 276 Bi. 6·51 : 6·94, 57″·4, 73° (1930). Common motion.

277 H and K lines in emission, +23 km/s.

280 γ Pav.

281 Mag. and col. not sufficiently observed to deserve standard weight.

289 θ PsA. Tri. AB 5·7 : 5·8, 0″·179, 35° (1954); AB, C 5·0 : 11, 35″·5, 339° (1918). Difficult spectrum, broad lines.

290 Bi. 6·6 : 9·7, 55″·0, 356° (1931).

291 Bi. 6·5 : 6·7, 25·8 *y*, $a = 0''{\cdot}237$, $p_d = 0''{\cdot}019$. All data for combined light. Difficult spectrum, broad lines.

294 ϵ Ind. H and K lines in emission, − 42 km/s.

296 α Gru. Bi. 1·7 : 12, 28″·5, 149° (1947). Mag. suspected of variability.

299 Bi. 5·4 : 10·0, 3″·2, 29° (1948).

300 ν Ind. Bi. 6·1 : 6·2, <0″·115 (1953). Not seen as Bi. since 1933. Duplicity suspect.

306 Mag. possibly variable.

307 H and K lines in emission, +5 km/s.

309 Bi. 6·0 : 10·0, 7″·8, 129° (1943).

310 Bi. 7·0 : 7·2, 27·2 *y*, a=0″·21, p_d=0″·015. All data refer to combined light. Mean date of vel. obs. 1954·07.

312 Difficult spectrum, broad lines.

313 This star has approximately the same parallax and proper motion as 314, but the rad. vel. differ by 10 km/s.

314 α PsA., Fomalhaut. Standard star in MKK Atlas.

318 Bi. 8·1 : 9·7, 3″·1, 107° (1935).

319 H and K lines in emission, +6 km/s.

320 θ Gru. Multiple star. AB 4·5 : 7·0, 1″·3, 68° (1955); AC 4·5 : 8·5, 161″, 293° (1904); CD 8·5 : 12, 95″, 335°. p_d (from AB)=0″·027. All data refer to combined light of AB. Mean date of vel. obs. 1953·12.

324 Bi. 7·3 : 9·5, 34″·3, 322° (1955).

325 Broad lines.

327 Bi. 6·9 : 8·7, 146 *y*, a=0″·656, p_d=0″·025.

330 μ Scl.

332 Bi. 8·2 : 8·7, 0″·8, 192° (1956). All data refer to combined light.

334 SX Phe This is the var. star with the shortest known period (80 min.). Maxima $7^m·10$–$6^m·54$: Minima $7^m·41$–$7^m·53$: Amplitude $0^m·8$–$0^m·3$. Absolute mag. too faint to be classed as RR Lyrae on present ideas. See J. Jackson (*MNASSA*, **8**, 29, 1949), O. J. Eggen (*P.A.S.P.*, **64**, 31 and 305, 1952), Th. Walraven (*BAN*, **12**, 57, 1953).

338 Bi. 6·8 : 7·3, 0″·36, 291° (1954).

339 θ Oct.

R 4 82 Eri.

R 7 Bi. 4·9 : 10·3, 2″·7, 341° (1928).

TABLE III

Relation between spectral type and colour index

Type	Cape			Johnson and Morgan		
	C.I.	No.	Range	C.I.	No.	Range
F0 III	+0·26	3	0·03			
K0 III	+1·05	5	0·14	+1·00	10	0·10
K1 III	+1·11	5	0·10	+1·12	2	0·03
K2 III	+1·23	3	0·13	+1·17	5	0·08
K3 III	+1·23	5	0·21	+1·30	4	0·15
K5 III	+1·34	4	0·10	+1·52	2	0·01
M0 III	+1·53	3	0·17			
F2 IV	+0·32	6	0·07	+0·35	1	
F5 IV	+0·41	9	0·12			
F6 IV	+0·42	4	0·06	+0·44	1	
G0 IV	+0·55	9	0·05	+0·61	2	0·06
G2 IV	+0·62	3	0·02			
G3 IV	+0·66	4	0·14			
G5 IV	+0·75	13	0·30			
G8 IV	+0·97	6	0·14	+0·86	1	
K0 IV	+1·07	12	0·29	+0·92	1	
F5 IV–V	+0·41	5	0·05	+0·40	1	
F6 IV–V	+0·48	3	0·06			
F8 IV–V	+0·48	7	0·03	+0·54	2	0·01
A3 V	+0·13	5	0·09	+0·09	3	0·03
A5 V	+0·24	3	0·02	+0·14	3	0·03
A7 V	+0·28	3	0·02	+0·18	3	0·01
F0 V	+0·34	7	0·09	+0·33	2	0·02
F2 V	+0·36	5	0·06	+0·38	3	0·03
F3 V	+0·42	3	0·04	+0·38	1	
F5 V	+0·45	11	0·05	+0·44	3	0·03
F6 V	+0·47	7	0·07	+0·46	4	0·04
F7 V	+0·49	3	0·06	+0·50	7	0·04
F8 V	+0·50	21	0·10	+0·53	3	0·04
G0 V	+0·56	18	0·14	+0·60	7	0·05
G1 V	+0·57	6	0·06	+0·61	1	
G2 V	+0·63	16	0·13	+0·64	2	0·01
G5 V	+0·71	21	0·16	+0·68	4	0·05
G8 V	+0·78	5	0·05	+0·74	2	0·03
K0 V	+0·88	15	0·15	+0·82	2	0·07
K2 V	+0·93	5	0·16	+0·88	2	0·02
K3 V	+0·94	6	0·20	+1·01	1	
K5 V	+1·12	11	0·13	+1·18	2	0·03

TABLE IV

Relation between spectral type and absolute visual magnitude

Type		M_v	No.	M_v		
				M and K	Gliese	Allen
G5	IV	3·7	4	3·3		3·1
F8	V	4·4	5	4·0		
G0	V	4·7	6	4·3	4·5	4·6
G1	V	4·9	5			
G2	V	4·8	8	4·6	4·7	
G5	V	5·2	11	5·0	5·0	5·3
G8	V	5·4	4	5·5	5·5	
K0	V	6·2	7	5·9	5·8	6·2
K2	V	6·3	4	6·3	6·3	
K3	V	6·8	5	6·8	6·5	
K5	V	7·1	10	7·7	7·2	7·6

Note

Following Johnson and Morgan (3), the M and K magnitudes have been adjusted by $-0^{m}\cdot 1$.

TABLE V

No.	$B-V$	M_V	π_{ph}	π_{tr}	μ_α	μ_δ	x	y	z	u	v	w
Type F3 V			″	″	″	″						
67	+0·40	3·65	0·015		+0·017	−0·055	+18	−53	−36	− 16	− 10	0
88	+0·41	3·73	0·020		−0·115	+0·020	0	−47	−17	+ 14	0	−25
166	+0·44	4·00	0·017		−0·092	−0·035	−45	−26	+28	− 6	− 35	+15
Type F5 V												
21	+0·45	4·08	0·017		+0·172	+0·020	−12	−30	−49	+ 40	− 28	+ 2
64	+0·47	4·24	0·022		−0·027	−0·121	−11	−37	−25	− 30	− 8	− 9
81	+0·45	4·08	0·062	0·041	−0·030	+0·386	+ 4	−15	− 5	+ 33	− 15	+ 1
97	+0·44	4·00	0·062	0·067	−0·200	+0·241	+ 6	−15	− 1	+ 31	− 17	− 4
172	+0·46	4·16	0·033	0·026	−0·194	−0·133	−25	−17	+ 6	+ 36	− 16	− 7
255	+0·42:	3·82	0·013		+0·061	−0·091	−64	− 7	−42	+ 53	− 24	+ 2
292	+0·45	4·08	0·032		−0·127	−0·063	−19	− 4	−25	− 30	− 10	− 7
316	+0·43	3·91	0·022		−0·018	−0·008	−20	− 6	−41	− 12	− 3	−16
337	+0·44	4·00	0·018		+0·109	+0·014	−16	−10	−52	+ 29	− 11	− 6
Type F6 V												
66	+0·45	4·08	0·021		+0·015	+0·164	+25	−32	−25	+ 38	+ 7	+ 5
168	+0·47	4·24	0·017		+0·038	−0·103	−45	−26	+28	− 36	− 26	−12
203	+0·45	4·08	0·025		+0·021	−0·108	−34	−20	− 8	+ 21	− 4	−12
231	+0·52	4·58	0·016		−0·134	+0·022	−58	− 8	−22	− 17	− 9	+36
248	+0·45	4·08	0·021		+0·081	+0·020	−40	− 4	−25	+ 14	+ 9	−12
303	+0·52	4·58	0·036	0·033	+0·245	000	−14	− 2	−24	+ 52	− 4	+27
Type F7 V												
17	+0·52	4·58	0·033		+0·165	+0·078	− 6	−14	−26	+ 29	+ 3	+15
158	+0·50	4·47	0·026		−0·057	−0·067	−27	−26	+10	+ 5	− 14	− 8
Type F8 V												
R 7	+0·46	4·16	0·070	0·047	−0·264	−0·169	− 8	−12	+ 1	+ 53	+ 41	−13
7	+0·54	4·68	0·017		−0·099	−0·096	− 8	−12	−57	− 39	− 11	−11
9	+0·47	4·24	0·021		+0·098	−0·046	−18	−29	−33	+ 5	− 34	−10
34	+0·55	4·73	0·021	0·058	−0·017	−0·280	+ 7	−23	−41	− 48	− 49	−16
45	+0·47	4·24	0·042		−0·023	−0·187	+ 6	−13	−19	− 15	− 18	−13
53	+0·52	4·58	0·027		+0·095	+0·083	+18	−18	−27	+ 21	− 6	+ 8
56	+0·45	4·08	0·039	0·026	+0·048	−0·270	+ 6	−18	−18	− 25	− 27	− 6
63	+0·50	4·47	0·044		−0·057	+0·273	− 4	−18	−13	+ 23	− 18	−26
65	+0·51	4·53	0·093	0·078	−0·042	+0·122	+ 1	− 9	− 6	+ 6	+ 4	− 1
108	+0·48	4·32	0·030		−0·089	+0·009	+ 1	−33	+ 4	+ 12	− 23	− 6
154	+0·51	4·53	0·036		−0·147	+0·023	−18	−20	+ 6	+ 29	+ 5	+ 2
188	+0·47	4·24	0·033		−0·093	−0·070	−28	−12	+ 2	+ 12	− 13	+ 2
217	+0·52	4·58	0·048		+0·134	−0·192	−20	− 4	− 3	+ 19	− 7	−18
220	+0·49	4·40	0·026		+0·031	−0·131	−37	− 8	− 9	+ 36	− 11	− 8
266	+0·53	4·63	0·022		+0·088	−0·007	−36	+10	−27	+ 37	− 6	+ 4
270	+0·48	4·32	0·017		+0·110	−0·075	−43	−18	−36	+ 28	− 16	−19
280	+0·47	4·24	0·100	0·111	+0·096	+0·807	− 6	− 4	− 7	+ 12	+ 47	+ 4
317	+0·53	4·63	0·023		−0·003	−0·124	−19	− 4	−39	− 21	− 27	−25
323	+0·49	4·40	0·056		+0·171	−0·025	− 8	− 7	−15	+ 15	− 8	− 5

TABLE V (*cont.*)

No.	$B-V$	M_V	π_{ph}	π_{tr}	μ_α	μ_δ	x	y	z	u	v	w
Type G0 V												
			″	″	″	″						
1	+0·58	4·83	0·029		+0·087	−0·084	−8	−5	−33	+6	−19	−2
20	+0·57	4·80	0·023		+0·119	−0·010	−9	−22	−37	+18	−15	+6
36	+0·58	4·83	0·028		−0·005	−0·110	+13	−12	−31	−15	−11	0
39	+0·57	4·80	0·020		−0·034	−0·114	−5	−33	−37	−29	−20	−12
70	+0·59	4·86	0·031	0·055	+0·094	+0·273	+11	−26	−16	+50	−23	+2
85	+0·53	4·63	0·023		+0·010	−0·120	+21	−37	−8	−12	−26	−12
94	+0·57	4·80	0·077	0·057	−0·293	+1·663	+5	−12	−1	+130	−68	+31
110	+0·52	4·58	0·084	0·073	−0·392	+0·241	+3	−11	+5	+32	−26	+8
125	+0·50	4·47	0·048		−0·083	+0·099	−2	−19	+9	+12	+3	+4
142	+0·50	4·47	0·035		−0·128	−0·092	−11	−25	+7	+13	−8	−15
193	+0·59	4·86	0·027	0·035	+0·054	−0·270	−35	−7	+10	+25	−24	−46
196	+0·57	4·80	0·031	0·030	−0·448	−0·191	−26	−18	−7	+2	−82	+23
204	+0·56	4·77	0·036		−0·166	−0·292	−23	−13	−5	−64	−85	−23
224	+0·52	4·58	0·044		+0·133	−0·162	−23	+2	−3	−5	−9	−22
229	+0·52	4·58	0·026		−0·017	−0·109	−36	−5	−13	−58	−29	−26
244	+0·55	4·73	0·033		−0·101	−0·081	−27	−2	−15	−34	−17	−6
274	+0·58	4·83	0·046	0·048	−0·532	−0·975	−16	−1	−15	−24	−103	+46
Type G1 V												
42	+0·58	4·83	0·082	0·081	+1·324	+0·664	−1	−8	−9	+70	−46	+19
59	+0·59	4·86	0·020	0·023	+0·265	−0·047	+11	−37	−32	+15	−85	+10
118	+0·58	4·83	0·049	0·049	−0·364	+0·056	+1	−19	+7	+33	−44	−1
159	+0·53	4·63	0·019		−0·090	−0·112	−38	−34	+14	+33	−11	−24
299	+0·59	4·86	0·078	0·077	+0·416	−0·662	−7	−3	−10	+27	−40	+9
309	+0·56	4·77	0·057	0·060	−0·012	−0·324	−9	−2	−15	−13	−28	−8
Type G2 V												
R 2	+0·56	4·77	0·128	0·134	+1·688	+1·163	−3	−4	−6	−66	0	−38
41	+0·63	4·92	0·075	0·105	+1·330	+0·669	−1	−9	−10	+77	−50	+21
127	+0·58	4·83	0·045	0·042	−0·517	−0·133	−3	−19	+11	+39	−34	−25
170	+0·66	5·00	0·020		−0·183	−0·070	−39	−30	+10	+70	−4	−6
176	+0·63	4·92	0·071	0·059	−1·625	−0·271	−12	−8	+2	+118	−43	+37
199	+0·68	5·10	0·028		+0·034	−0·107	−35	−6	+6	−13	−12	−14
211	+0·66	5·00	0·025		−0·010	−0·125	−40	−4	+1	+20	−19	−12
241	+0·59	4·86	0·035		+0·008	−0·265	−26	−4	−11	+39	−39	+4
251	+0·63	4·92	0·059	0·047	+0·837	−0·681	−12	−8	−9	+69	−29	−45
269	+0·63	4·92	0·016		−0·061	−0·200	−48	−3	−40	−23	−61	−2
282	+0·65	4·97	0·030		+0·208	−0·073	−21	+8	−25	+41	−19	−3
290	+0·59	4·86	0·071	0·068	+0·142	−0·297	−9	−2	−11	+11	−20	+2
305	+0·62	4·91	0·025	0·013	+0·459	+0·009	−17	+7	−36	+55	−10	−81
324	+0·69	5·15	0·037	0·035	+0·216	−0·150	−12	−10	−22	+11	−36	−17
Type G3 V												
19	+0·69	5·15	0·023	0·068	+0·309	−0·277	+5	−4	−43	+27	−91	−76
198	+0·63	4·92	0·037		+0·055	−0·100	−25	−10	0	−59	−28	−14

TABLE V (*cont.*)

No.	$B-V$	M_V	π_{ph}	π_{tr}	μ_α	μ_δ	x	y	z	u	v	w
Type G5 V			″	″	″	″						
R 4	+0·70	5·20	0·155	0·156	+3·047	+0·744	+1	−4	−5	+83	−95	−21
29	+0·72	5·30	0·062	0·091	−0·230	+0·452	+5	−3	−15	+14	+35	−12
68	+0·77	5·64	0·055	0·040	−0·147	−0·093	+1	−15	−10	−8	−4	−18
71	+0·77	5·64	0·046	0·027	−0·118	−0·471	+6	−18	−11	−43	−14	−23
79	+0·71	5·25	0·109	0·115	+0·121	−0·218	−2	−8	−4	−18	−29	−12
95	+0·74	5·43	0·040	0·007	−0·064	+0·275	−1	−24	−7	+31	−31	0
103	+0·74	5·43	0·044	0·042	−0·002	−0·286	+4	−22	+1	−22	−9	−20
139	+0·64	4·94	0·102	0·092	−1·538	+0·393	−3	−9	+4	+65	−39	+6
153	+0·77	5·64	0·020		−0·176	−0·164	−32	−37	+12	+29	−41	−27
174	+0·61	4·90	0·038		+0·022	−0·110	−22	−14	+5	+4	−3	−14
177	+0·70	5·20	0·028		−0·185	−0·020	−32	−12	+12	+1	−28	+21
215	+0·71	5·25	0·040		−0·224	−0·181	−24	−6	−3	+28	−26	+14
216	+0·70	5·20	0·105	0·071	−0·021	−0·197	−9	−3	−2	+13	−4	−1
261	+0·73	5·36	0·057	0·051	−0·022	−0·532	−14	+3	−10	−4	−43	−7
268	+0·67	5·05	0·042		−0·027	−0·200	−18	+5	−14	+11	−27	+10
278	+0·70	5·20	0·053	0·038	+0·483	−0·429	−12	−6	−13	+54	−29	−6
297	+0·65	4·97	0·026		+0·177	−0·062	−22	0	−32	+20	−17	−24
311	+0·73	5·36	0·032	0·038	−0·369	−0·209	−14	−1	−28	−63	−16	+9
321	+0·69	5·15	0·027		+0·251	+0·090	−17	−12	−31	+44	+1	−17
Type G6 V												
182	+0·70	5·20	0·068	0·069	−0·428	−0·212	−13	−6	+3	+19	−27	+6
Type G7 V												
84	+0·77	5·64	0·031	0·065	−0·035	−0·178	+3	−30	−12	−20	−35	−27
Type G8 V												
209	+0·80	5·85	0·119	0·125	+1·010	+0·190	−8	−3	−1	−29	+21	−31
254	+0·75	5·50	0·247	—	+1·194	−1·145	−3	−2	−2	+37	−6	−5
			—	0·170			−4	−3	−3	+47	−12	−13
308	+0·80	5·85	0·049	0·056	+0·363	+0·034	−9	+2	−18	+39	−8	+1
326	+0·78	5·71	0·027	0·041	+0·196	−0·300	−13	−7	−34	+16	−59	+11
Type K0 V												
2	+0·95	6·63	0·054	0·014	+0·287	+0·111	−2	+1	−18	+26	−3	−9
27	+0·81	5·92	0·091	0·083	+2·142	+0·662	0	−5	−10	+101	−76	−25
32	+0·94	6·59	0·074		+0·128	+0·045	+2	−6	−12	+14	−25	−37
48	+0·89	6·35	0·075	0·060	+0·213	+0·286	+4	−7	−11	+31	−16	−17
86	+0·91	6·45	0·088	0·063	+0·198	+0·052	+6	−9	−2	+6	−14	+8
132	+0·80	5·85	0·095	0·099	−0·679	+0·818	−3	−9	+5	+52	+20	+15
135	+0·84	6·10	0·026		−0·169	−0·017	−14	−35	+8	+27	−10	−11
143	+0·82	5·98	0·059	0·070	−0·537	−0·306	−8	−15	−2	+37	−19	−26
189	+0·80	5·85	0·046	—	−0·872	−1·397	−18	−12	−2	+83	−142	−42
			—	0·063			−13	−9	−1	+58	−105	−31
200	+0·91	6·45	0·065		−0·104	−0·033	−14	−6	−1	−25	−17	+3
206	+0·89	6·35	0·062	0·079	+0·081	+0·600	−14	−8	−4	−33	+29	+18
259	+0·91	6·45	0·140	0·116	+1·247	−0·213	−6	+2	−4	+68	−12	−9
281	+0·94:	6·59	0·067	0·063	+0·249	+0·191	−10	−1	−11	+6	+12	−20

TABLE V (cont.)

No.	$B-V$	M_V	π_{ph}	π_{tr}	μ_α	μ_δ	x	y	z	u	v	w
Type K1 V			″	″	″	″						
102	+0·93	6·55	0·100	0·091	−0·312	+0·347	+ 2	−10	+ 1	+ 24	− 7	0
227	+0·97	6·71	0·097		+0·008	−0·130	−10	+ 1	− 2	− 24	− 3	− 8
Type K2 V												
28	+1·02	6·88	0·092		+0·094	−0·016	+ 3	− 2	−10	+ 13	− 12	−36
52	+0·91	6·45	0·037	0·023	+0·195	+0·630	+ 7	−16	−20	+ 87	+ 8	− 7
284	+0·89	6·35	0·069	0·069	+0·424	−0·208	− 9	− 2	−11	+ 5	− 22	−37
295	+0·86	6·21	0·035	0·049	+0·377	−0·258	−17	− 2	−23	+ 52	− 40	− 4
Type K3 V												
11	+0·94	6·59	0·077	0·103	+0·615	+0·040	0	− 1	−13	+ 33	− 18	+12
161	+1·04	6·92	0·113	0·098	−0·455	−0·816	− 6	− 6	0	+ 32	− 9	−26
187	+0·84	6·10	0·062	0·071	−0·130	+0·326	−14	− 9	− 1	− 39	− 13	+22
256	+0·87	6·26	0·154	0·174	+0·442	−1·555	− 6	+ 1	− 3	+118	− 53	+49
328	+1·01	6·85	0·089	0·102	+0·115	−0·737	− 5	− 6	− 8	− 31	− 71	−24
Type K5 V												
33	+1·05	6·94	0·038	0·013	+0·067	−0·516	+ 3	−13	−23	− 42	− 60	−11
44	+1·16	7·09	0·063	0·060	+0·363	−0·241	− 2	−11	−11	− 8	− 63	−18
55	+1·14	7·03	0·076	0·069	+0·763	+0·401	+ 2	− 9	− 9	+ 36	− 6	+46
112	+1·18	7·15	0·079	0·087	+0·446	−0·433	0	−13	+ 2	− 37	+ 9	− 6
137	+1·07	6·98	0·069	0·075	−0·660	+0·241	− 5	−13	+ 4	+ 41	− 27	+ 8
141	+1·16	7·09	0·106	0·096	−1·075	−0·627	− 3	− 7	+ 5	+ 18	− 70	− 5
208	+1·16	7·09	0·143	0·172	−0·479	−1·143	− 7	0	+ 1	− 1	− 40	− 9
294	+1·06	6·96	0·279	0·285	+3·941	−2·569	− 2	− 1	− 3	+ 79	− 40	+ 6
301	+1·10	7·00	0·073	0·070	+0·226	−0·809	− 7	+ 2	−12	− 7	− 56	−16
307	+1·14	7·03	0·068	0·065	+0·375	−0·027	− 6	+ 2	−13	+ 21	− 8	−13
313	+1·13	7·02	0·129	0·123	+0·321	−0·171	− 3	+ 1	− 7	+ 2	− 8	−18
Type K7 V												
222	+1·31	7·74	0·075	0·073	+0·122	−0·438	−13	− 3	− 3	+ 10	− 20	−18
Type M0 V												
258	+1·44	8·94	0·157	0·158	+0·778	−0·152	− 5	− 1	− 4	+ 38	+ 2	− 3
277	+1·42	8·74	0·258	0·255	−3·283	−1·146	− 3	0	− 3	− 58	− 20	+28
Type M2 V												
319	+1·50	9·54	0·273	0·273	+6·786	+1·300	− 1	0	− 3	+106	− 14	−55

In addition to those given in Table V, trigonometrical parallaxes from (**9**) and (**10**) are available for the following stars :

No.	π	No.	π	No.	π
	″		″		″
4	0·153	117	0·061	205	0·014
5	30	122	15	207	183
6	70	124	53	212	15
22	62	131	85	230	11
23	148	138	6	232	20
24		145	6	234	48
25	6	147	29	235	
26	10	148	38	243	24
37	32	150	− 4	247	10
40	70	151	7	249	12
49	42	157	18	252	− 15
51	− 1	162	32	260	35
54	37	163	33	267	17
57	38	165	15	272	43
58	59	167	49	275	12
76	29	169	6	276	− 6
83	35	171	24	279	12
89	69	178	10	289	2
92	50	179	53	296	51
96	1	180	5	300	30
98	19	181	9	306	9
99	46	183	78	314	144
104	29	186	15	320	12
105	7	190	22	330	21
109	59	191	25	333	12
113	23	197	27	334	23
115	0·011	201	0·021	339	0·011

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THE FORMATION OF POPULATION I STARS
PART I. GRAVITATIONAL CONTRACTION

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Summary

The present paper is preliminary to the second part in which it will be suggested that the gravitational contraction that is primarily significant for the formation of Population I stars is that of the material destined to form a cluster of such stars. The relevant type of gravitational instability is that recently studied by Ebert and by Bonnor, which concerns material under external pressure. A more elementary treatment than theirs is here developed from a general form of the virial theorem. This shows that the essential features of their results hold good under somewhat relaxed conditions which may be more like those encountered in applications. But also it shows that if we employ the exact results of Ebert and Bonnor we may have to regard them as applying to somewhat larger masses (up to about fifty per cent larger) than those to which they explicitly refer, on account of incomplete realization of the theoretical conditions. The present treatment appears to afford some additional physical insight into the results obtained by Ebert and Bonnor and their relation to other criteria for gravitational instability. This is aided by its extension to cylindrical and plane-stratified distributions, which had been shown by Ebert to provide instructive comparisons. Among more detailed results we exhibit the behaviour for varying temperature of isothermal gravitating material under given external pressure and we note some numerical results required for subsequent use.

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1. *Introduction*.—A theory of the formation in interstellar matter of condensations that might lead to the production of fresh stars has been discussed in an important paper by Ebert (1955). In Part II of the present work it will be suggested that a condensation of the sort immediately considered by Ebert comprises the material of a cluster, rather than of a star, but that the stars of the cluster may be produced by a repetition of the same basic process. Bonnor (1956) has also studied the process independently, though with regard to a different astronomical application. The object of this paper is to review the process and to obtain some numerical results for subsequent use.

Ebert and Bonnor investigate the gravitational instability of a mass of gas under an applied pressure. The type of instability discovered by them is likely always to be more significant than the types previously discussed by other authors. The reasons for this are stated more particularly by Bonnor. Certainly, it appears to be the type that is relevant to the applications here in view. However, the results given by the authors are those for an unstable state that is approached through a sequence of states of strict equilibrium. Before using the results for the proposed applications, we need to be satisfied that they remain valid, to a sufficiently good approximation, under less idealized conditions. The somewhat elementary discussion about to be given is presented in confirmation of this.

Also it appears to give useful physical insight into the phenomenon as well as a better understanding of the employment of the virial theorem in this connection.

2. *A general form of the virial theorem.*—Consider a body of gas that can be regarded as permanently composed of the same set of molecules or "particles". Let there be a Newtonian frame such that there is no mass-motion of the gas relative to this frame at the epoch of interest, say t_0. Let a typical particle of mass m have position vector $\mathbf{r}$ in the frame at time t, and let $\mathbf{P}$ be the force acting upon it. Its equation of motion is

$$m\ddot{\mathbf{r}} = \mathbf{P}$$

and we have

$$\frac{d^2}{dt^2}(\mathbf{r}\,.\,\mathbf{r}) = 2\frac{d}{dt}(\mathbf{r}\,.\,\dot{\mathbf{r}}) = 2\dot{\mathbf{r}}^2 + 2\mathbf{r}\,.\,\ddot{\mathbf{r}}.$$

Hence, summing over all particles of the set,

$$\frac{1}{2}\frac{d^2}{dt^2}(\Sigma m\mathbf{r}^2) = \Sigma m\dot{\mathbf{r}}^2 + \Sigma\mathbf{r}\,.\,\mathbf{P}. \tag{2.1}$$

We have $\Sigma mr^2 = I$, the moment of inertia about the origin. At t_0 we have $\Sigma\frac{1}{2}m\dot{\mathbf{r}}^2 = \mathscr{T}$, the total thermal kinetic energy, since there is then no mass-motion. The quantity $\Sigma\mathbf{r}\,.\,\mathbf{P}$ is the virial. The summation in the virial is by definition over all particles of the set. But this is equivalent to summation over all forces acting in the system such that, if $\mathbf{P}$ be now regarded as a typical force, $\mathbf{r}$ is the position vector of its point of application.

We assume that the only forces are those acting in collisions of the particles with each other or with particles of any other material present, together with the gravitational attraction upon the particles exerted by the other particles of the system or by other material present.

The forces in collisions between particles of the system make no contribution to the virial because they consist of equal and opposite pairs having the same point of application. If the forces in collisions with other material may be treated as producing a pressure p at a boundary S, then their contribution is $\int_S p\mathbf{r}\,.\,\mathbf{dS}$, where $\mathbf{dS}$ has the direction of the inward normal. If $\rho\,dv$ is the mass of the particles in a volume-element dv at $\mathbf{r}$, and if $\mathbf{g}$ is the total gravitational force per unit mass at $\mathbf{r}$, then the rest of the virial is $\int_V \rho\mathbf{r}\,.\,\mathbf{g}\,dv$, where V is the volume enclosed by S. Thus (2.1) becomes

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2\mathscr{T} + \int_S p\mathbf{r}\,.\,\mathbf{dS} + \int_V \rho\mathbf{r}\,.\,\mathbf{g}\,dv \qquad (t=t_0). \tag{2.2}$$

This is a form of the virial theorem somewhat more general than that usually quoted (Jeans, 1925; Chandrasekhar, 1939).

Equation (2.2) applies to a body of gas released at time t_0 from rest in a given configuration and under given external pressure. So at t_0 we have

$$dI/dt = 0 \qquad (t=t_0). \tag{2.3}$$

If the given state is an equilibrium state we must also have

$$d^2I/dt^2 = 0 \qquad (t=t_0), \tag{2.4}$$

and so the vanishing of the right-hand side of (2.2) gives a necessary condition for equilibrium. The condition is not, of course, sufficient. But if (2.3), (2.4) are satisfied, we shall call the state a state of *quasi-equilibrium* in the present context. Otherwise, if the given initial conditions are such that (2.2) yields $d^2I/dt^2 > 0$, < 0, we take this as a criterion that the gas as a whole begins to expand

or contract, respectively. This is the criterion generally used in such work. But it has to be noted that (2.2) will not apply further to the motion itself, since once the expansion or contraction is under way this will constitute mass-motion which is not taken into account in (2.2). However, the equation may show that there is no equilibrium state to which the motion could lead, in which case we are able to conclude that the expansion must proceed indefinitely or that the contraction leads to collapse.

If the material has uniform temperature T then

$$2\mathcal{T} = 3M\mathfrak{R}T/\mu, \tag{2.5}$$

where M is the total mass of the material concerned and μ is its mean molecular weight (where $\mu = 1$ for atomic hydrogen).

If p has the same value at all points of S then

$$\int_S p\mathbf{r}\,.\,\mathbf{dS} = p\int_S \mathbf{r}\,.\,\mathbf{dS} = -p\int_V \operatorname{div}\mathbf{r}\,dv = -3pV. \tag{2.6}$$

If $\mathbf{g}$ arises solely from the self-gravitation of the material concerned then (Chandrasekhar, 1939)

$$\int_V \rho\mathbf{r}\,.\,\mathbf{g}\,dv = \Omega \tag{2.7}$$

where Ω (<0) is the potential energy compared with a state of complete dispersal.

If all these conditions are satisfied, (2.2) becomes

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3M\mathfrak{R}T/\mu - 3pV + \Omega \qquad (t = t_0). \tag{2.8}$$

For equilibrium or quasi-equilibrium this gives

$$pV = M\mathfrak{R}T/\mu + \tfrac{1}{3}\Omega. \tag{2.9}$$

In the case of equilibrium, if the last term is negligible, this reproduces Boyle's law. If Ω is not negligible, (2.9) expresses the appropriate modification of Boyle's law. But we need to know more about the equilibrium state in order actually to evaluate Ω.

3. *Isolated body of isothermal gas: temperature given.*—We here confine attention to material at a given uniform temperature T, so that (2.5) holds good with the same value of T for any different states of the same material.

We consider a body of gas in equilibrium under no forces other than its self-gravitation and a uniform pressure p over its boundary. This pressure is assumed to be exerted by other gases of the same or a different species. Then equation (2.8) is applicable. But also, because of the way in which the pressure is applied, the equilibrium state must possess spherical symmetry (*cf.* Lyttleton, 1953). Therefore, on account of its physical dimensions, Ω is here of the form

$$\Omega = -A\gamma M^2/R, \tag{3.1}$$

where γ is the gravitational constant, R is the radius of the configuration, and A is a positive dimensionless number depending upon the density distribution.

The quantity A is the same for homologous states of bodies of different mass. It is not necessarily the same for different states of compression of the same mass. But on general grounds we expect it to vary only slowly with R for equilibrium states and *we shall provisionally neglect any dependence of A upon R.* We shall verify later that this is in fact a good approximation.

Using (3.1), we find from (2.9)

$$p = \frac{3M(\mathfrak{R}T/\mu)}{4\pi R^3} - \frac{A\gamma M^2}{4\pi R^4}. \tag{3.2}$$

This is the modified form of Boyle's law for the present case.

It is convenient to define quantities R_1, p_1, ρ_1 having the dimensions of length, pressure, density, by

$$R_1 = \frac{\gamma M}{(\mathfrak{R}T/\mu)}, \qquad p_1 = \frac{(\mathfrak{R}T/\mu)^4}{4\pi\gamma^3 M^2}, \qquad \rho_1 = \frac{(\mathfrak{R}T/\mu)^3}{4\pi\gamma^3 M^2}. \tag{3.3}$$

Then (3.2) becomes

$$\frac{p}{p_1} = 3\left(\frac{R_1}{R}\right)^3 - A\left(\frac{R_1}{R}\right)^4. \tag{3.4}$$

We are considering the case of given M, T, i.e. given R_1, p_1. If R is sufficiently great, the second term on the right in (3.4) is small compared with the first. If then the pressure is slowly increased, so that the system may be regarded as traversing equilibrium states, R will decrease. It is seen from (3.4) how the proportionate effect of gravitation in assisting the compression increases as R decreases. In fact, *there is a maximum value of the pressure for which any equilibrium state is possible.* This behaviour is illustrated by Fig. 1, curve UU.

The maximum occurs at $R = R_{cr}$ where, from (3.4),

$$R_{cr}/R_1 = 4A/9, \tag{3.5}$$

and the critical pressure is given by

$$p_{cr}/p_1 = \tfrac{3}{4}(4A/9)^{-3}. \tag{3.6}$$

The corresponding value of the mean density $\bar{\rho}$ is given by

$$\bar{\rho}_{cr}/\rho_1 = 3(4A/9)^{-3}. \tag{3.7}$$

We may notice that this is four times the density that would be produced by the same pressure without the assistance of gravitation, in other words the density that a small specimen of the gas would have in the same surroundings. For the unmodified Boyle's law in the present units is simply $p/p_1 = \rho/\rho_1$.

If *the pressure applied at the boundary is increased beyond the critical value p_{cr} the system must collapse under its own gravitation.* This is evident from the fact that if $p > p_{cr}$ equation (3.4) cannot be satisfied for any value of R. But it is also confirmed by going back to (2.8) and observing that if $p > p_{cr}$ then $d^2I/dt^2 < 0$ for all R, i.e. the system cannot remain at rest in any state but must always contract. This conclusion does not depend upon the assumption that A is approximately constant; it follows from the fact that A has a positive lower bound.

We see from (3.4) that for p less than the critical value there are equilibrium states having $R < R_{cr}$; these are shown by the dotted part of curve UU in Fig. 1. But these states are unstable with respect to gravitational collapse. For, if the radius is slightly decreased below the equilibrium value, the presssure remaining constant, (2.8) gives $d^2I/dt^2 < 0$ so that the contraction proceeds further, and there is no other equilibrium state in which it can be halted. If we retain the approximation of treating A as constant, even such unstable states exist only in $R_{\min} < R < R_{cr}$, where $R_{\min}$ is the value of R for which $p = 0$ in (3.4), i.e.

$$R_{\min}/R_1 = \tfrac{1}{3}A. \tag{3.8}$$

For any smaller value of R the material would collapse under its own gravitation without the assistance of any external pressure, as can again be confirmed from (2.8). In fact, *we have here the familiar condition for collapse that is inferred from the virial theorem in the form*

$$2\mathcal{T}+\Omega<0 \qquad (p=0). \tag{3.9}$$

But this more usual application of the virial theorem cannot be so significant as the preceding one. For the system cannot reach the point of collapse from $R_{\min}$ without first being brought to the point of collapse from R_{cr}. The essential physical difference is even more obvious from the cases considered in Section 10.

Bonnor (1955, pp. 353, 357) mentions briefly some consequences of the usual virial theorem, but he is not concerned to pursue them. Part of our present purpose is to show what sort of conclusions may legitimately be drawn from a suitable form of the theorem. Broadly speaking, we find it to give useful results so long as these are viewed as a whole. But it may be misleading to draw isolated inferences from it such as taking (3.9) as a generally significant condition for gravitational instability.

It is to be emphasized that the discussion in this section concerns material that is maintained at a given temperature. The question as to whether this is the case required for applications will arise in Part II.

4. *Isolated body of isothermal gas: pressure given.*—The properties of the critical state naturally do not depend upon its being approached through states of the same temperature. By way of illustration, we may consider instead states for which the applied pressure is given.

Equation (3.2) may be rewritten as

$$\frac{\mathbf{R}T}{\mu}=\frac{4\pi pR^3}{3M}+\frac{A\gamma M}{3R}. \tag{4.1}$$

We write

$$R_2=(4\pi)^{-1/4}\gamma^{1/4}p^{-1/4}M^{1/2}, \qquad \mathbf{R}T_2/\mu=(4\pi)^{1/4}\gamma^{3/4}p^{1/4}M^{1/2},$$
$$\rho_2=(4\pi)^{-1/4}\gamma^{-3/4}p^{3/4}M^{-1/2}. \tag{4.2}$$

Then (4.1) becomes

$$\frac{T}{T_2}=\frac{1}{3}\left(\frac{R}{R_2}\right)^3+\tfrac{1}{3}A\frac{R_2}{R}. \tag{4.3}$$

The right-hand side of (4.3) has a minimum at $R=R_{cr}$ where

$$R_{cr}/R_2=(A/3)^{1/4} \tag{4.4}$$

for which the value of T is T_{cr} where

$$T_{cr}/T_2=\tfrac{4}{3}(A/3)^{3/4} \tag{4.5}$$

leading to

$$\bar{\rho}_{cr}/\rho_2=3(A/3)^{-3/4}. \tag{4.6}$$

It is easily verified that (3.5), (4.4); (3.6), (4.5); (3.7), (4.6) agree in giving the same critical state.

The immediate interpretation here is that, given M, p, *there is a minimum temperature below which no equilibrium state is possible.*

The general interpretation of the work is that (3.6) or (4.5), which may be written

$$p_{cr}=\frac{3^7}{2^8A^3}\frac{(\mathbf{R}T_{cr}/\mu)^4}{4\pi\gamma^3M^2}, \tag{4.7}$$

gives the locus of critical states in the T, p-diagram so that for given M there is no equilibrium state for any pair of values of T, p corresponding to a point above this locus (see Fig. 4).

5. *Gravitational collapse.*—Returning to the behaviour of material at a given temperature T, we have seen that for any value of this temperature, and for any value of the pressure p, the material will certainly collapse if it starts with a small enough radius. Now a point we have sought to make is that, so long as the material behaves in accordance with the modified Boyle's law, it cannot reach such a small radius without first being brought to a critical point on (4.7) from which it would collapse anyhow. Nevertheless the question arises as to whether the material must inevitably obey the law—must it climb to the peak of the curve UU (Fig. 1) or can it in any way slip through? The answer appears to be that the gas must obey the law so long as there is no singularity in its density and that a singularity can arise only after the system as a whole has attained a critical state.

However, this conclusion applies directly only to the case we have been considering of an isolated body of gas. If there are gravitating bodies immersed in it these can act as condensation nuclei (in the manner of Bondi's symmetric accretion (Bondi, 1952; McCrea, 1956) although the steady-approximation may not be adequate). Thus there appears to be no permanent equilibrium state in this case. But in the proposed applications of the theory the time-scale for the effect of condensation nuclei to become significant is almost certainly long compared with the times of other changes. So the effect may be ignored in these applications.

6. *Exact theory (Ebert).*—Before making further use of the results of Section 3, we recall Ebert's exact solution of the same problem. This is, of course, obtained by solving the equation of hydrostatic equilibrium under the appropriate boundary conditions. The equation reduces to Emden's equation for an isothermal distribution and the problem is that of adapting Emden's solution so as to exhibit the equilibrium of a given mass of material under a given applied pressure.

Let $\zeta(\xi)$ be the solution of the Emden equation

$$\frac{d^2\zeta}{d\xi^2}+\frac{2}{\xi}\frac{d\zeta}{d\xi}+e^{\zeta}=0 \tag{6.1}$$

such that $\zeta(0)=0$, $\zeta'(0)=0$. Then the properties of a mass M under pressure p are expressed in terms of ξ as parameter and the quantities R_1, p_1, ρ_1 of (3.3) as follows:

Applied pressure	$p=p_1e^{\zeta}(\xi^2\zeta')^2,$	(6.2)
Radius	$R=R_1(-\xi\zeta')^{-1},$	(6.3)
Central density	$\rho_{cent}=\rho_1(\xi^2\zeta')^2,$	(6.4)
Mean density	$\bar{\rho}=3\rho_1(-\xi\zeta')^3,$	(6.5)
Surface density	$\rho=\rho_1e^{\zeta}(\xi^2\zeta')^2,$	(6.6)
Potential energy	$\Omega=M(\mathfrak{R}T/\mu)(-3-e^{\zeta}\xi\zeta'^{-1}),$	(6.7)
Virial coefficient	$A=3(-\xi\zeta')^{-1}-e^{\zeta}\zeta'^{-2}.$	(6.8)

Expressions (6.2), (6.3) were given by Ebert. The others are readily obtainable from his work: in particular the expression (6.7) for Ω follows most easily by using (2.9), and then (6.8) follows by using (3.1). It is to be emphasized

that to any particular value of ξ there corresponds a particular configuration that can exist under the applied pressure given by (6.2) and that the remaining quantities characterize the configuration as a whole; the formulae do *not* express the variation of any quantity through the configuration. It may be noted that small values of ξ give small values of ζ and $-\zeta'$ and consequently small values of p and large values of R.

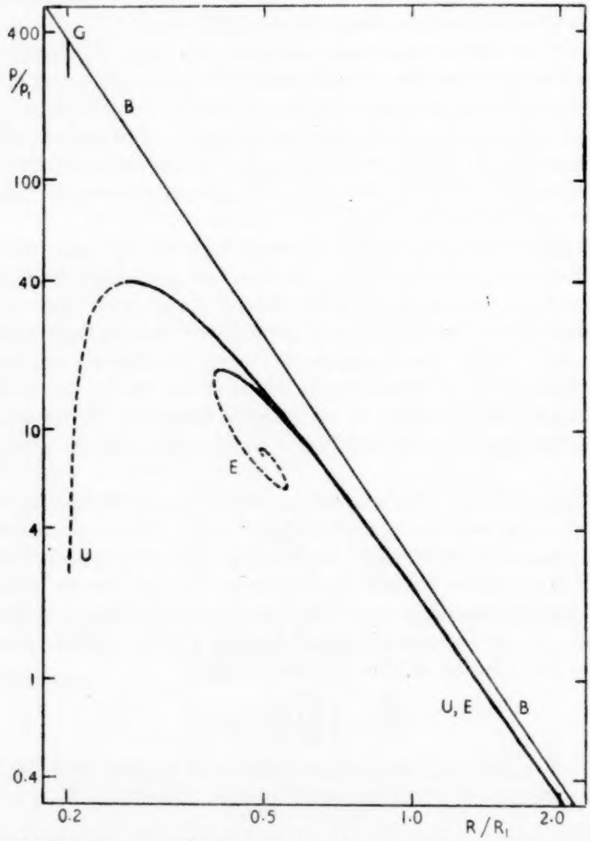

FIG. 1.—*Relation between radius R and external pressure p for a given mass of gas at given temperature :* EE *according to solution of Emden equation (exact theory of Ebert and Bonnor);* UU *for uniform density according to the approximate theory of the present paper;* BB *according to Boyle's law neglecting gravitation. The critical points on* EE *and* UU *are the maxima on these curves; the broken parts represent states that could not be attained in practice; the point G on* BB *corresponds to the radius from which gravitational contraction would occur according to a common interpretation of the virial theorem. (Scales are logarithmic.)*

The behaviour resulting from these formulae is obtained by using Emden's (1907) numerical solution for $\zeta(\xi)$. In the significant range of configurations it is found to be qualitatively like that predicted by the elementary theory in Section 3 (and we shall see later that it is also quantitatively very similar). This is shown by the curves drawn by Ebert and by Bonnor and the authors' discussion of these curves. Part of the p, R curve is reproduced in Fig. 1; a few numerical

values are given in Table I, chiefly to show the approach to the critical state given by the last line.

TABLE I

Isothermal sphere, temperature T, under external pressure p

ξ	T fixed		p fixed		$\rho_{cent}/\bar{\rho}$	A
	p/p_1	R/R_1	T/T_2	R/R_2		
2	2·04	1·06	0·837	1·27	1·41	0·616
4	13·0	0·506	0·527	0·961	2·70	0·666
6	17·5	0·418	0·489	0·855	5·02	0·720
6·5	17·6	0·410	0·488	0·840	5·78	0·733

This exact theory again shows the existence of a critical state corresponding to that given by (3.5), (3.6). For given M, T, the critical pressure is that given by the smallest value of ξ ($\neq 0$) for which $dp/d\xi = 0$, that is from (6.2)

$$e^{\zeta}\xi^4\zeta'(2\zeta'' + \zeta'^2 + 4\xi^{-1}\zeta') = 0.$$

Using (6.1), this gives

$$\zeta'^2 - 2e^{\zeta} = 0. \tag{6.9}$$

From Emden's numerical solution of (6.1) we find that (6.9) yields approximately

$$\xi = 6{\cdot}5, \qquad e^{\zeta} = 0{\cdot}070, \qquad \zeta' = -0{\cdot}375. \tag{6.10}$$

Hence we obtain from (6.2)–(6.8) the critical values, approximately,

$$R_{cr} = 0{\cdot}41 R_1, \tag{6.11}$$

$$p_{cr} = 17{\cdot}6 p_1, \tag{6.12}$$

$$\bar{\rho}_{cr} = 43{\cdot}4 \rho_1, \tag{6.13}$$

$$(\rho_{cent}/\bar{\rho})_{cr} = 5{\cdot}8, \qquad (\rho_{cent}/\rho)_{cr} = 14{\cdot}3, \tag{6.14}$$

$$A_{cr} = 0{\cdot}733. \tag{6.15}$$

The values (6.11), (6.12) agree with those given by Ebert.

From (6.12) the relation between critical values of p_{cr} for given M, corresponding to (4.7) is

$$p_{cr} = 17{\cdot}6 \frac{(\mathfrak{R} T_{cr}/\mu)^4}{4\pi\gamma^3 M^2}. \tag{6.16}$$

Using (6.15) in (4.7) would give a coefficient 21·7 in place of 17·6; this is some indication of the error introduced by treating A as constant, as distinct from using an approximation to A_{cr} itself. As another illustration (3.5) would give 0·33 in place of the 0·41 in (6.11).

We also note that the elimination of M between (6.11), (6.12) gives

$$R_{cr}^2 p_{cr} = 2{\cdot}96 \frac{(\mathfrak{R} T/\mu)^2}{4\pi\gamma}. \tag{6.17}$$

Finally, for given M, p and using the definitions (4.2), we find

$$R = R_2 (e^{\zeta} \zeta'^{-2})^{1/4} \tag{6.18}$$

$$T = T_2 [e^{\zeta}(\xi^2 \zeta')^2]^{-1/4}. \tag{6.19}$$

These together give in parametric form the relation corresponding to (4.3). The

relation is shown in graphical form in Fig. 2. As before, these results lead to the same critical states as those got from given M, T.

7. *Approximate theory: the parameter A.*—The theory given in sections 3-5 would be exact if we were to use the value of A given by the exact calculations of equilibrium states. But in that case there would be no object in having the former theory at all; the results of section 6 would supply everything we need. We are going, however, to make wider use of the former theory.

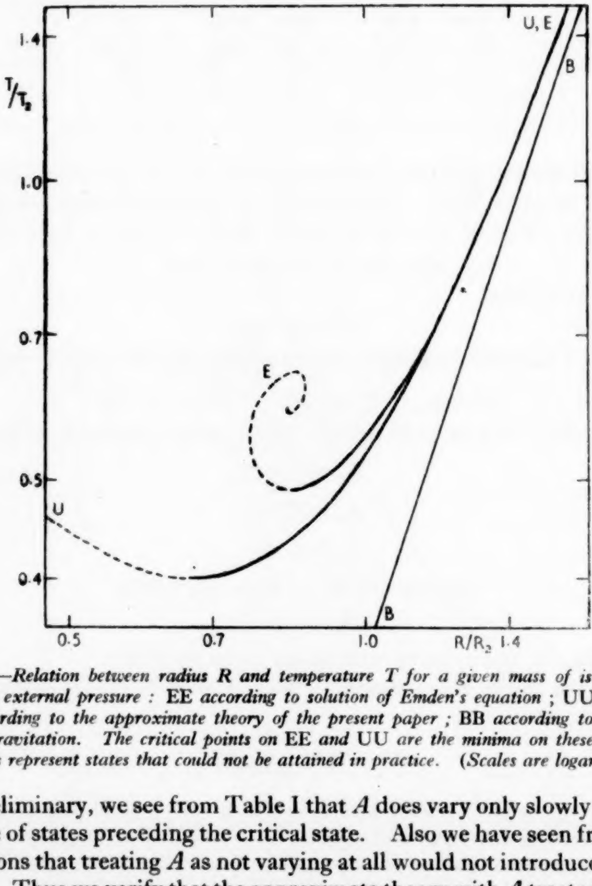

FIG. 2.—*Relation between radius R and temperature T for a given mass of isothermal gas under given external pressure:* EE *according to solution of Emden's equation;* UU *for uniform density according to the approximate theory of the present paper;* BB *according to Boyle's law neglecting gravitation. The critical points on* EE *and* UU *are the minima on these curves; the broken parts represent states that could not be attained in practice.* (*Scales are logarithmic.*)

As a preliminary, we see from Table I that A does vary only slowly in the significant range of states preceding the critical state. Also we have seen from a couple of illustrations that treating A as not varying at all would not introduce a considerable error. Thus we verify that the approximate theory with A treated as constant would have given a reliable first approximation in circumstances in which we do in fact know the exact results.

Now in order for collapse to occur as calculated according to the exact theory at some particular values of p_{cr}, T_{cr}, just before collapse the system must be in the corresponding equilibrium state. In particular it must have developed the degree of concentration of its mass towards the centre indicated by equations (6.14). But we wish to apply the theory to a cloud of interstellar matter acted upon by external pressure arising in ways we shall describe. The requirements which we shall envisage occur in what must be regarded as rather rapidly changing conditions.

So it would be an over-idealization to suppose that, if gravitational collapse is produced, it is preceded by the establishment of a strict equilibrium state. Rather than to suppose the cloud to possess the degree of central condensation required for this, it would be more realistic to ignore the possibility of systematic density-gradients and to treat the density as being approximately uniform.

We are interested here only in a cloud that does eventually collapse. In the first place this means, as we shall see, that the cloud must be roughly spherical.

In the second place, we should not apparently be giving the process over-favourable treatment if we suppose the compression, prior to collapse, to proceed through a sequence of quasi-equilibrium states. For we thereby ignore the mass-motion associated with the compression and this mass-motion is in the sense that would actually assist ultimate collapse.

For these various reasons we shall now write down results corresponding to those of sections 3-5, *taking the system to be merely a uniform sphere of gas satisfying* (2.9) *regarded as the equation of quasi-equilibrium*. For the reasons stated, these results may be more significant for applications than those of the exact theory. But we shall see that, considering all the approximations involved in the applications, the differences between the two sets of results are not great. Thus we may also conclude that the exact theory is not misleading for the required applications in spite of its high degree of idealization.

8. *Uniform sphere.*—The coefficient A for a uniform sphere is $3/5$. In this case therefore the relations (3.5)-(3.7), (4.4)-(4.7) become approximately (the mean density being here the uniform density)

$$R_{cr} = 0{\cdot}27\, R_1 \quad (= 0{\cdot}40\, R_1^*) \tag{8.1}$$

$$p_{cr} = 39{\cdot}5\, p_1 \quad (= 17{\cdot}6\, p_1^*) \tag{8.2}$$

$$\bar{\rho}_{cr} = 158\, \rho_1 \quad (= 70\, \rho_1^*) \tag{8.3}$$

$$R_{cr} = 0{\cdot}67\, R_2 \quad (= 0{\cdot}82\, R_2^*) \tag{8.4}$$

$$T_{cr} = 0{\cdot}40\, T_2 \quad (= 0{\cdot}49\, T_2^*) \tag{8.5}$$

$$\bar{\rho}_{cr} = 10\, \rho_2 \quad (= 8{\cdot}2\, \rho_2^*) \tag{8.6}$$

$$p_{cr} = 39{\cdot}5\, \frac{(\mathfrak{R} T_{cr}/\mu)^4}{4\pi\gamma^3 M^2} \left(= 17{\cdot}6\, \frac{(\mathfrak{R} T_{cr}/\mu)^4}{4\pi\gamma^3 M^{*2}} \right). \tag{8.7}$$

The curve UU in Fig. 1 is drawn for the case of a uniform sphere. It is seen that the curve closely follows that for true equilibrium states up to nearly the critical point for the latter. But its own critical point occurs for a somewhat higher pressure and smaller density. This is because the compressibility of the true equilibrium configurations is assisted by the concentration of material towards the centre. However, the difference is not such as to be of much importance in the type of application we wish to make.

In this case the elimination of M between (8.1), (8.2) gives

$$R_{cr}^2 p_{cr} = 2{\cdot}81\, \frac{(\mathfrak{R} T/\mu)^2}{4\pi\gamma}. \tag{8.8}$$

The coefficient here is nearly the same as in (6.17). Consequently if we take a particular value of p_{cr} here it will cause a mass M to collapse from a radius R_{cr}, and in the exact theory it will cause a mass M^* to collapse *from practically the same radius*. We find that a good approximation is

$$M^* = \tfrac{3}{2} M. \tag{8.9}$$

This is shown by the fact that the values in brackets in (8.1)-(8.7) are got by replacing M by M^* given by (8.9) in the definitions of R_1, etc. so giving R_1^*, etc. It is seen that the coefficients of R_1^*, p_1^* in (8.1), (8.2) are almost identical with those of R_1, p_1 in (6.11), (6.12).

It is therefore of interest to compare the p, R graphs for a mass $\frac{2}{3}M$ according to the exact theory of section 6 and a mass M according to the theory of the present section. These are shown in Fig. 3 where it has seemed more natural not to use logarithmic scales as in Figs. 1, 2. We see that in the region of interest, that is as the critical point is approached by compression, the curves are very similar. The inference to be drawn from this is that *in applications we may use the exact theory while estimating that the departure of actual conditions from the conditions of the theory could require the masses concerned to be increased by up to about fifty per cent.*

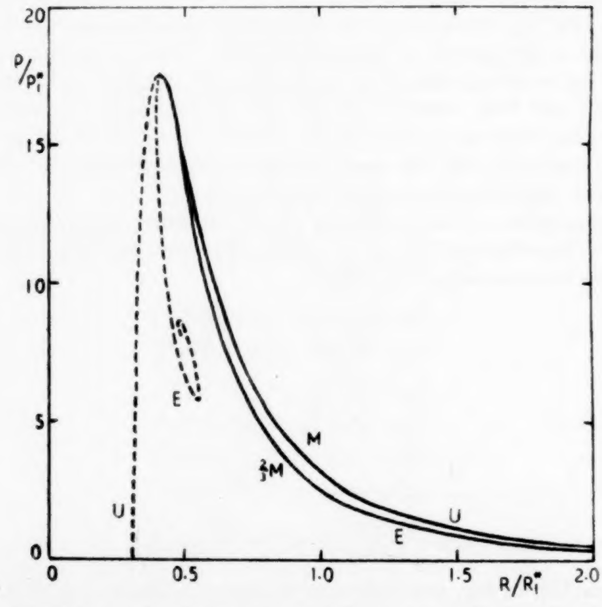

FIG. 3.—*Comparison of p-R relations for mass* $\frac{2}{3}M$ *according to solution of Emden's equation* (EE) *and mass M for uniform density according to approximate theory* (UU).

9. *Critical values.*—Returning to Ebert's results, if for given T, p we take the value M_{cr}, say, of M determined by (6.16), then at temperature T any mass exceeding M_{cr} will be caused to collapse by external pressure p.

If we substitute for M_{cr} in terms of T, p in (6.11) we obtain the critical radius R_{cr} for this mass at the point of collapse. We can assert that no quantity of the material at temperature T under external pressure p can exist in equilibrium so as to have radius exceeding R_{cr}.

If we substitute for M_{cr} in terms of T, p in (6.13) we obtain the critical mean density $\bar{\rho}_{cr}$ for this mass at the point of collapse. We can assert that no quantity of the material at temperature T under external pressure p can exist in equilibrium so as to have a mean density exceeding $\bar{\rho}_{cr}$. We find, in fact, that

$$\bar{\rho}_{cr} = 2{\cdot}46p(\mathfrak{R}T/\mu)^{-1} \qquad (9.1)$$

so that the greatest departure from Boyle's law that can be produced with the assistance of gravitation before actual gravitational collapse gives a density nearly $2\frac{1}{2}$ times the density in the absence of appreciable gravitation. (The elementary theory gave from (3.6), (3.7) the factor 4.)

Following Ebert, we use for interstellar matter the value

$$\mu = 1 \cdot 5. \tag{9.2}$$

(The present writer previously used the value 1·4 (McCrea 1953, 1956).) Then the numerical results may be conveniently expressed as

$$M_{cr} = 8 \cdot 91\, T^2 (p/k)^{-1/2} \text{ solar masses} \tag{9.3}$$

$$R_{cr} = 2 \cdot 85\, T (p/k)^{-1/2} \text{ parsecs} \tag{9.4}$$

$$\bar{n}_{cr} = 2 \cdot 46\, T^{-1} (p/k) \text{ particles cm}^{-3} \tag{9.5}$$

where k is Boltzmann's constant and $\bar{n}_{cr}$ is the particle-density corresponding to $\bar{\rho}_{cr}$. Some illustration of the values given by (9.3) is shown graphically in Fig. 4 mainly for the purposes of Part II. This figure serves also as the graph of (6.16), and qualitatively of (4.7).

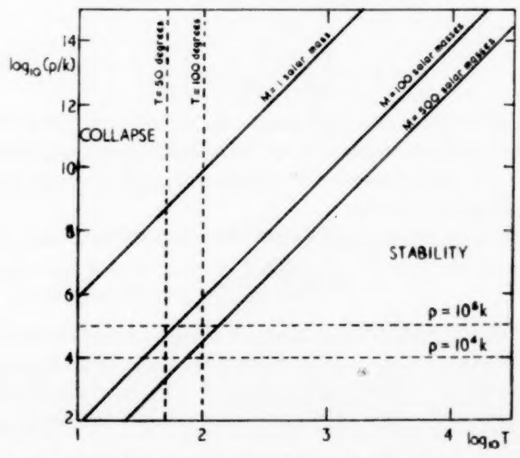

FIG. 4.—*Relation between critical temperature and critical pressure for given mass M according to solution of Emden's equation* ($M = 1$, $M = 100$, $M = 500$ *solar masses*).

For comparison, the criterion (3.8) or (3.9) applied to uniform material of n particles cm^{-3} and using (9.2) gives a critical mass and radius

$$M_{cr} = 41 \cdot 1\, T^{3/2} n^{-1/2} \text{ solar masses} \tag{9.6}$$

$$R_{cr} = 6 \cdot 4\, T^{1/2} n^{-1/2} \text{ parsecs.} \tag{9.7}$$

10. *Other distributions.*—Ebert obtained instructive results by considering distributions other than the spherically symmetric one. We shall treat these by the elementary theory of section 2 for comparison with his work.

Cylindrical distribution. Consider a cylindrical distribution of gas in equilibrium under no forces other than its own gravitation and a uniform pressure p over its boundary, this pressure being exerted by further gas of the same or a different species. All properties are supposed not to vary in the direction of the generators of the cylinder, which is therefore regarded as of effectively infinite

length. For equilibrium there must be cylindrical symmetry. Let s be the radius of the boundary and m the mass of material per unit length of the cylinder.

We apply the virial theorem to the material in a length l of the cylinder between right sections. Equation (2.5) holds good with M replaced by ml. The first equality in (2.6) no longer holds good because the pressure is not uniform over the plane sections. But we may take it as defining a mean pressure p', say, over the whole boundary of the region, so that the final member of (2.6) becomes $-3\pi s^2 lp'$. Equation (2.7) is not applicable because the field $\mathbf{g}$ arises partly from the rest of the cylinder outside the region considered. But from dimensional considerations we have

$$\int_V \rho\mathbf{r}\cdot\mathbf{g}\,dv = -B\gamma m^2 l \tag{10.1}$$

where B is a positive dimensionless number depending upon the density distribution. As in the case of the coefficient in (3.1), we provisionally neglect any dependence of B upon s.

The virial theorem (2.2) then becomes for equilibrium or quasi-equilibrium

$$0 = 3ml(\mathfrak{R}T/\mu) - 3\pi s^2 lp' - B\gamma m^2 l$$

giving

$$p' = \left(\frac{\mathfrak{R}T}{\mu} - \tfrac{1}{3}B\gamma m\right)\frac{m}{\pi s^2} \tag{10.2}$$

If we ignore gravitation the second term on the right of (10.2) is absent and the pressure over the whole boundary is p; we should have again the standard form of Boyle's law. Taking account of gravitation, (10.2) gives the modified form of Boyle's law for the present case.

We see from (10.2) that no equilibrium state is possible at any radius if

$$m \geqslant \frac{3(\mathfrak{R}T/\mu)}{B\gamma}. \tag{10.3}$$

Otherwise, whatever the value of p', there is always an equilibrium state for which the radius s is given by (10.2). This follows from the boundedness of B without assuming its constancy. *Thus in the case of a cylindrical distribution the material can never be brought to the point of gravitational collapse by compression at any given temperature.*

Like (3.8), the condition (10.3) is equivalent to the familiar condition (3.9). But we now see how the fact that (3.9) happens to give the only condition for gravitational collapse in one case, the present one, is no indication that it is the significant condition in other cases, e.g. the spherically symmetric case. A similar remark applies to the use of the criterion for gravitational instability obtained by Jeans.

Ebert's treatment. Ebert gave a concise rigorous proof that

$$p \propto s^{-2}$$

corresponding to (10.2). His work shows that in this case the various states of compression are homologous so that actually B is constant and p' is a constant multiple of p.

However, Ebert appears not to have remarked upon the existence of an upper bound for m for equilibrium, that is upon the property associated with (10.3). This property arises because a cylindrical distribution of gas at given temperature in equilibrium under its own gravitation alone, and with no singularity in the

density along the axis, is of unique *finite* mass per unit length. The mass that can be in equilibrium under an applied pressure must be less than this amount. If we attempt to add mass in excess of this amount, the system will collapse.

This accounts for the difference from the spherical case. For there the distribution in equilibrium under its own gravitation and with no central singularity is of infinite mass. Consequently, *any* finite mass of sufficiently large radius can be held in equilibrium by an applied pressure. However, in that case, the various states of compression are not homologous and so, as we have seen, gravitational collapse will occur when the radius is made sufficiently small.

The writer is not aware of any published proof of the stated property of the cylindrical distribution. In Ebert's notation, where ξ is proportional to the radial distance from the axis and η is proportional to the density at ξ, the appropriate "Emden" equation is

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\frac{\xi}{\eta}\frac{d\eta}{d\xi}\right) = -\eta. \tag{10.4}$$

Also the mass within distance ξ is proportional to

$$-\frac{\xi}{\eta}\frac{d\eta}{d\xi}. \tag{10.5}$$

Making the substitutions

$$\xi = e^{\tau}, \quad \xi^2\eta = e^{-\chi},$$

(10.4) becomes

$$d^2\chi/d\tau^2 = e^{-\chi} \tag{10.6}$$

and the expression (10.5) becomes

$$2 + d\chi/d\tau. \tag{10.7}$$

If we suppose that χ exists as $\tau \to \infty$, then it can be seen from (10.6) that the only consistent possibilities for χ, $d\chi/d\tau$ are

$$\chi \to \infty, \; d\chi/d\tau \to \text{constant} \quad (\tau \to \infty).$$

Hence either the distribution does not tend to infinity or the expression (10.7) tends to a finite limit. This is sufficient to establish the required property.

Possibility of break-up. We wish to make some estimate of the stability of the distribution against longitudinal disturbances. It will be adequate here to use the approximation of treating the undisturbed density ρ as uniform because a radial variation of ρ will not have a direct effect on the problem.

We have then

$$m = \pi s^2 \rho \tag{10.8}$$

At a point distant q from the axis of the cylinder $\mathbf{g}$ is directed towards the axis and

$$|\mathbf{g}| = 2\gamma m q/s^2 \quad (q < s) \tag{10.9}$$

It is easily seen that the integral in (10.1) is independent of the choice of origin of $\mathbf{r}$. If we take this to be a fixed point on the axis, the projection of $\mathbf{r}$ upon the direction of $\mathbf{g}$ is $-q$, so

$$\mathbf{r} \cdot \mathbf{g} = -2\gamma m q^2/s^2$$

Thus (10.1) becomes

$$-\int_V \rho \mathbf{r} \cdot \mathbf{g}\, dv = 2\gamma m \rho (l/s^2) \int_0^s 2\pi q^3\, dq = \pi\gamma m \rho l s^2 = \gamma m^2 l,$$

using (10.8), so that in this approximation

$$B = 1. \tag{10.10}$$

In this case (10.3) may be written, again using (10.8),

$$s^2 \geqslant \frac{3}{\pi} \frac{(\mathfrak{R}T/\mu)}{\gamma\rho} \tag{10.11}$$

[For comparison, (3.8) leads in the case of uniform density to the analogous condition

$$R^2 \geqslant \frac{15}{4\pi} \frac{(\mathfrak{R}T/\mu)}{\gamma\rho} \tag{10.12}$$

Thus, as might be expected, (10.11), (10.12) give roughly the same critical distance.]

Now the Jeans criterion (Bonnor, 1956, equation (3.7)) gives the condition for gravitational instability for isothermal disturbances of wavelength λ

$$\lambda^2 > \pi \frac{(\mathfrak{R}T/\mu)}{\gamma\rho} \tag{10.13}$$

If we apply this to disturbances propagated *along* the cylindrical distribution, it indicates that any tendency for the cylinder to break up by gravitational action would produce portions having length of the order of λ given by (10.13). But (10.13) is derived from the theory of propagation of plane waves in an infinite medium. It can be a significant approximation in the present case only if the lateral extent is at any rate comparable with λ. However, (10.11) shows that the medium could not then be in equilibrium but must in fact be collapsing laterally.

On the other hand, if the radius is considerably less than the lower bound of s given by (10.11) and, consequently, than the lower bound of λ given by (10.13), then the gravitational efficiency of longitudinal disturbances would be reduced. Thus the significant lower bound of λ would be even greater than that given by (10.13). Hence any gravitational break-up of the cylinder in this case could result only in portions that are themselves still of length large compared with their radius.

Combining all the results of this section we conclude that a long uniform filament of intersteller material, provided it can withstand any lateral pressure at all, can withstand any amount of such pressure and that, moreover, it is gravitationally stable against break-up into short portions. Had the latter part of the conclusion not been reached, the first part by itself would not possess much significance.

Plane-stratified distribution. Consider a plane stratified layer of gas in equilibrium under no forces other than its own gravitation and a uniform pressure over its two parallel plane boundaries. All properties are supposed not to vary in any direction parallel to these planes. For equilibrium, the distribution must be symmetrical about a median plane. Let $2z$ be the thickness of the layer and let σ be the mass per unit area. We apply the virial theorem to the material inside a volume $l \times l \times 2z$ of the layer. As in the previous case, we may define a mean pressure p' over the boundary such that the pressure term in (2.2) is again $-3p' \times$ volume, that is $-6p'l^2z$ in this case. Analogously to (10.1) we have

$$\int_V \rho\mathbf{r} \cdot \mathbf{g}\, dv = -C\gamma\sigma^2 l^2 z, \tag{10.14}$$

where C is dimensionless. Then corresponding to (10.2) we obtain

$$0 = 3\sigma l^2(\mathfrak{R}T/\mu) - 6l^2zp' - C\gamma\sigma^2 l^2 z$$

giving

$$p' = \sigma \frac{(\mathfrak{R}T/\mu)}{2z} - \tfrac{1}{6}C\gamma\sigma^2. \tag{10.15}$$

Ignoring the gravitational term in (10.15) we again get the elementary form of Boyle's law. But we also see in this case, unlike that of a spherical distribution, how even allowing for gravitation the material behaves more closely in accordance with this law the greater the compression, i.e. the smaller the value of z. This is because for an infinite plane-stratified distribution the gravitational attraction at distance q, say, from the median plane depends only upon the amount of material between $\pm q$, and not upon the value of q itself. Therefore gravitational effects are not increased by compression, whereas the thermal pressure is increased.

Interpreting (10.15) as in previous cases, this equation suffices to show that there must be a value of z yielding an equilibrium state for any applied pressure. Then the material can never be brought to the point of collapse by compression, as we should also expect from the argument just given.

In this case *there is no possibility at all of gravitational collapse* of the sort we have seen to be possible for a spherical distribution. This again illustrates the untrustworthiness of attempting to argue from the behaviour of gravitational effects in one dimension to that in three dimensions. The possibility of genuine gravitational *collapse*, as distinct from some sort of contraction from an unrealizable initial state, seems to have been first made clear in Ebert's discussion of the three-dimensional case.

An exact solution for the present plane-stratified case was also given by Ebert. It confirms the inferences we have drawn from the virial treatment. Actually the latter may not be pushed much further than we have taken it because it is not a good approximation to ignore the variation of the coefficient C over a large range of z.

11. *General interpretation.*—Interstellar matter is often observed in the form of elongated distributions and probably also in "sheets". It is sometimes remarked that these give the appearance of being under compression by other parts of the material. We now see from Ebert's work and the present discussion that gravitational collapse cannot occur directly from these forms. In any case, the distributions that are most commonly recognized as filamentary or sheet-like are not thought to have any immediate connection with star formation. Also their behaviour is not likely to be much influenced by self-gravitation. The only interest of the present work in relation to such distributions is that it confirms their gravitational stability even if their self-gravitation is significant.

Nevertheless, the bodies of gas concerned in star-formation probably arise from temporary distributions that have either an elongated form, as the "elephant trunks" considered by Ebert, or a flattened form, as the compressed regions considered by Oort (1955). These undoubtedly break up quite rapidly. But their break-up must be owing to irregularities in the density-distribution and in the external pressure to which it is subjected, or to differential motions. Our discussion, based on Ebert's work, shows that gravitational effects cannot play a significant part at this stage. It shows that it is only *after* such break-up has produced a body of gas of roughly the same diameter in all directions that gravitational collapse may ensue. Whether it does so or not can then be determined by Ebert's and Bonnor's theory for spherical distributions. The present work is claimed to show that the use of their results is not vitiated by a lack of exact realization of the conditions under which they are obtained.

Since this was written, a paper by Mestel and Spitzer (1956) on star formation has been published. While their paper is concerned mainly with magnetic effects which are not dealt with here, it does make extensive use of the virial theorem and conditions for gravitational contraction derived therefrom. Those conditions are open to the criticisms mentioned in section 3 above, although any consequent amendment would probably not substantially affect the authors' conclusions.

A preliminary account of the present work was given during the spring of 1956 in lectures in the Berkeley Astronomical Department of the University of California where the writer was privileged to hold a visiting professorship and also a Fulbright travel grant.

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1957 *April* 20.

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RADIAL VELOCITIES OF SOUTHERN B STARS DETERMINED AT THE RADCLIFFE OBSERVATORY*

(PAPER II)

M. W. Feast, A. D. Thackeray and A. J. Wesselink

Summary

In the second stage of the Radcliffe programme on radial velocities of southern O and B type stars, velocities of 130 stars selected for distance (chiefly $m_0-M \geqslant 11 \cdot 0$) are presented together with 59 stars with previously observed velocities. All the stars have been classified on the MK system.

The fainter stars demanded the use of faster emulsions compared with Paper I, and an investigation of residuals of the lines He 4471, 4026 has shown that for these emulsions significant deviations from IAU wave-lengths occur which vary with line-width, in accordance with Petrie's findings. The Radcliffe procedure as regards wave-lengths of 4471, 4026 has been revised accordingly.

The stars of this paper with newly determined velocities are at an average distance more than twice as great as that of the stars in Paper I.

*The full text of this paper is published in *Memoirs of the R.A.S.*, **68,** Part I, 1957.

NOTICE TO AUTHORS

1. *Communications.*—Papers must be communicated to the Society by a Fellow. They should be accompanied by a summary at the *beginning* of the paper conveying briefly the content of the paper, and drawing attention to important new information and to the main conclusions. The summary should be intelligible in itself, without reference to the paper, to a reader with some knowledge of the subject; it should not normally exceed 200 words in length. **Authors are requested to submit MSS. in duplicate. These should be typed using double spacing and leaving a margin of not less than one inch on the left-hand side. Corrections to the MSS. should be made in the text and not in the margin.** Unless a paper reaches the Secretaries more than seven days before a Council meeting it will not normally be considered at that meeting. By Council decision, MSS. of accepted papers are retained by the Society for one year after publication; unless their return is then requested by the author, they are destroyed.

2. *Presentation.*—Authors are allowed considerable latitude, but they are requested to follow the general style and arrangement of *Monthly Notices*. References to literature should be given in the standard form, including a date, for printing either as footnotes or in a numbered list at the end of the paper. Each reference should give the **name and initials** of the author cited, irrespective of the occurrence of the name in the text (some latitude being permissible, however, in the case of an author referring to his own work). The following examples indicate the style of reference appropriate for a paper and a book, respectively:—

A. Corlin, *Zs. f. Astrophys.*, **15**, 239, 1938.
H. Jeffreys, *Theory of Probability*, 2nd edn., section 5.45, p. 258, Oxford, 1948.

3. *Notation.*—For technical astronomical terms, authors should conform closely to the recommendations of Commission 3 of the International Astronomical Union (*Trans. I.A.U.*, Vol. VI, p. 345, 1938). Council has decided to adopt the I.A.U. 3-letter abbreviations for constellations where contraction is desirable (Vol. IV, p. 221, 1932). In general matters, authors should follow the recommendations in *Symbols, Signs and Abbreviations* (London: Royal Society, 1951) except where these conflict with I.A.U. practice.

4. *Diagrams.*—**These should be designed to appear upright on the page, drawn about twice the size required in print and prepared for direct photographic reproduction except for the lettering, which should be inserted in pencil. Legends should be given in the manuscript indicating where in the text the figure should appear.** Blocks are retained by the Society for 10 years; unless the author requires them before the end of this period they are then destroyed.

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9. Postcards giving the programme of each meeting are issued some days before the meeting concerned. Fellows wishing to receive such cards whether for Ordinary Meetings or for the Geophysical Discussions or both should notify the Assistant Secretary

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